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CHARACTERIZATION OF ELASTIC SOLIDS USING FINITE ELEMENT METHODS

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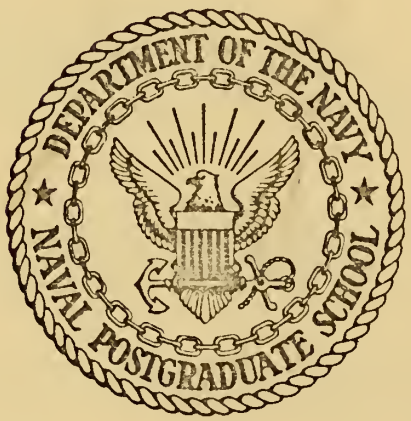

THESIS

CHARACTERIZATION OF ELASTIC SOLIDS
USING
FINITE ELEMENT METHODS

by

Brian Alfred Edwards

Thesis Advisor: G. Cantin

December 1972

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Characterization of Elastic Solids Using Finite Element Methods

by

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Lieutenant, United States Navy
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Submitted in partial fulfillment of the
requirements for the degree of

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ABSTRACT

Finite element methods are applied to the problem of characterizing linear, anisotropic elastic solids. The conventional finite element displacement formulation is used to simulate an elastic material in plane stress. An inverted finite element formulation is then applied, and the characterizing six material constants are calculated as numerical results.

A possible test device for the experimental characterization of anisotropic solids is postulated, the precision of displacement measurements to be required for such a device being determined by random perturbation analysis. Numerical constants accurate to within three percent are predicted if a precision of one part in eight hundred (1/800) can be measured. Numerical constants accurate to within one percent are predicted if a precision of one part in eight thousand (1/8000) can be measured in the test device.

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I. INTRODUCTION

Jaensson and Sundström [11] used stress and strain approximations, developed by a finite element analysis of the microstructure of a presumed isotropic WC-Co alloy, as source data from which Young's modulus and Poisson's ratio were calculated as the characterizing constants for the alloy. A more direct application of the finite element method to the characterization of elastic solids is possible and can be applied to a wider range of materials. When finite element characterization is applied to isotropic solids an immediate advantage is secured in that Young's modulus and Poisson's ratio are obtained simultaneously. When the finite element characterization technique is applied to a general linear, anisotropic elastic material an overwhelming advantage is obtained in that no other method provides an experimental determination of all six anisotropic material constants.

The ability to characterize general linear, anisotropic elastic materials makes the design of a test device incorporating the finite element characterizing technique highly desireable. The application of this technique to linear elastic problems has been largely ignored, although an increasing use of the technique has been made to problems in the mechanical characterization of physically and kinematically non-linear materials [6, 7, 9, 12, 13, 15, 16].

The work reported in this study concerns a general class of linear, anisotropic materials. Its goal is to identify and determine preliminary design parameters applicable to the design of a testing device to be used in the characterization of linear, anisotropic solids. The finite

element technique employed in this characterization scheme has been observed [12] to demonstrate a sensitivity to the measurements of displacement data, and it is a specific goal of this study to identify such sensitivities and the degree of measurement precision required.

The study has been divided into three parts. In the first part the analytics of the finite element characterization are examined. Rather tedious derivations result in closed-form formulations, around which the characterization technique is built. The second part of the study uses a structural analysis computer code to simulate necessary test displacement data. The third and final part of the study required the writing of a computer program incorporating the finite element analytics. Perturbations of the test displacement data and consequent computer solutions to the characterization problem permitted a determination of the measurement precisions required for the possible test device.

II. THEORETICAL BACKGROUND

A. THEORY OF ELASTICITY

All structural materials exhibit in various degrees the property of elasticity: external forces loading a structure produce deformations of the structure, and if these forces do not exceed a limiting value the deformations disappear when the forces are removed. The theory of elasticity provides mathematical relations between forces and displacements acting in a structure. The action of forces lead to the definition of the stress tensor, and the geometric deformations lead to the definition of the strain tensor. A relation between these two tensors is called a constitutive law [8, 10, 14, 19]. It is this law, in its simplest formulation, which is paramount to the characterization of linear elastic solids.

1. Generalized Linear, Anisotropic Materials in Three-dimensions

The general form of the constitutive law in the theory of elasticity, a generalized Hooke's law, is the set of functions relating stresses to strains:

$$\begin{aligned}
\sigma_x &= f_1(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) \\
\sigma_y &= f_2(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) \\
\sigma_z &= f_3(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) \\
\tau_{xy} &= f_4(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) \\
\tau_{xz} &= f_5(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) \\
\tau_{yz} &= f_6(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})
\end{aligned} \tag{1}$$

where x, y, z are the axes of the orthogonal, Cartesian coordinate system; σ_x, σ_y, σ_z are the normal stresses; τ_{xy}, τ_{xz}, τ_{yz} are the

shear stresses; ε_{xx}, ε_{yy}, ε_{zz} are the normal strains; and γ_{xy}, γ_{xz}, γ_{yz} are the shear strains.

In the case of small deformations the simplest form of functions (1) is a linear set of equations. Denoting the coefficients applicable to this linear formulation by a_{mn}:

$$\begin{aligned}
\sigma_x &= a_{11}\varepsilon_{xx} + a_{12}\varepsilon_{yy} + a_{13}\varepsilon_{zz} + a_{14}\gamma_{xy} + a_{15}\gamma_{xz} + a_{16}\gamma_{yz} \\
\sigma_y &= a_{21}\varepsilon_{xx} + a_{22}\varepsilon_{yy} + a_{23}\varepsilon_{zz} + a_{24}\gamma_{xy} + a_{25}\gamma_{xz} + a_{26}\gamma_{yz} \\
\sigma_z &= a_{31}\varepsilon_{xx} + a_{32}\varepsilon_{yy} + a_{33}\varepsilon_{zz} + a_{34}\gamma_{xy} + a_{35}\gamma_{xz} + a_{36}\gamma_{yz} \\
\tau_{xy} &= a_{41}\varepsilon_{xx} + a_{42}\varepsilon_{yy} + a_{43}\varepsilon_{zz} + a_{44}\gamma_{xy} + a_{45}\gamma_{xz} + a_{46}\gamma_{yz} \\
\tau_{xz} &= a_{51}\varepsilon_{xx} + a_{52}\varepsilon_{yy} + a_{53}\varepsilon_{zz} + a_{54}\gamma_{xy} + a_{55}\gamma_{xz} + a_{56}\gamma_{yz} \\
\tau_{yz} &= a_{61}\varepsilon_{xx} + a_{62}\varepsilon_{yy} + a_{63}\varepsilon_{zz} + a_{64}\gamma_{xy} + a_{65}\gamma_{xz} + a_{66}\gamma_{yz}
\end{aligned} \tag{1a}$$

Using matrix notation, the set of functions (1a) is written:

$$\{\sigma\} = [A]\{\varepsilon\} \tag{1b}$$

or,

$$\{\varepsilon\} = [B]\{\sigma\} \tag{1c}$$

where $\{\sigma\}$ and $\{\varepsilon\}$ are the vectors whose elements are, respectively, the stress and strain components of the stress and strain tensors at a point,

$\{\sigma\} = [\sigma_x,\sigma_y,\sigma_z,\tau_{xy},\tau_{xz},\tau_{yz}]^T$ and $\{\varepsilon\} = [\varepsilon_{xx},\varepsilon_{yy},\varepsilon_{zz},\gamma_{xy},\gamma_{xz},\gamma_{yz}]^T$;

and $[A]$ and $[B]$ are symmetric, 6×6 matrices of coefficients, $[A]^{-1} = [B]$, consisting of twenty-one different constants, a_{mn}.

Restricting attention to elasticity in two-dimensions, two distinct problem types are identified: plane stress problems, requiring

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0,$$

and plane strain problems, requiring

$$\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0.$$

2. Elasticity in Plane Stress

In the plane stress problem the in-plane stresses are defined as:

$$\{\sigma^P\} = [\sigma_x, \sigma_y, \tau_{xy}]^T.$$

Corresponding to these in-plane stresses are in-plane strains:

$$\{\varepsilon^P\} = [\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}]^T.$$

Other strain components also exist. The out-of-plane strain components are defined as:

$$\{\varepsilon^{OP}\} = [\varepsilon_{zz}, \gamma_{xz}, \gamma_{yz}]^T,$$

where $\{\varepsilon^{OP}\}$ is the vector of these out-of-plane strains.

The constitutive equation for the plane stress problem is, in terms of equation (1b):

$$\begin{Bmatrix} \sigma^P \\ 0 \end{Bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* \\ A_{12}^{*T} & A_{22}^* \end{bmatrix} \begin{Bmatrix} \varepsilon^P \\ \varepsilon^{OP} \end{Bmatrix}, \tag{2}$$

where

$$A_{11}^* = \begin{bmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{bmatrix}; \quad A_{12}^* = \begin{bmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{bmatrix}; \quad A_{22}^* = \begin{bmatrix} a_{33} & a_{35} & a_{36} \\ a_{53} & a_{55} & a_{56} \\ a_{63} & a_{65} & a_{66} \end{bmatrix};$$

and the a_{mn} are the linear coefficients in equation (1b).

From equation (2):

$$\{\varepsilon^{OP}\} = -[(A^*_{22})^{-1}(A^*_{12})^T]\{\varepsilon^P\} \tag{2a}$$

and,

$$\{\sigma^P\} = [A^*_{11}]\{\varepsilon^P\} + [A^*_{12}]\{\varepsilon^{OP}\}$$

$$= [A^*_{11} - (A^*_{12})(A^*_{22})^{-1}(A^*_{12})^T]\{\varepsilon^P\} . \tag{2b}$$

In the plane stress problem the stress components only are in-plane: the state of strain is three-dimensional with the out-of-plane components given by equation (2a).

Alternately, equation (2b) is written:

$$\{\sigma^P\} = [A']\{\varepsilon^P\} \tag{2c}$$

where $[A']$ is a symmetric, 3 x 3 matrix of constants:

$$[A'] = [A^*_{11} - (A^*_{12})(A^*_{22})^{-1}(A^*_{12})^T],$$

the determination of which characterizes a linear, anisotropic material in plane stress.

For perfectly isotropic materials submitted to a state of plane stress:

$$A'_{11} = \frac{E}{1-\nu^2} = A'_{22}$$

$$A'_{12} = \nu A'_{11}$$

$$A'_{13} = A'_{23} = 0 \tag{3}$$

$$A'_{33} = \frac{1}{2}(1-\nu)A'_{11}$$

where E and ν are Young's modulus and Poisson's ratio, respectively, and A'_{11}, A'_{12}, A'_{13}, A'_{22}, A'_{23}, and A'_{33} are the six elastic constants forming $[A']$.

If an isotropic material with $E = 29.50 \times 10^3$ ksi and $\nu = 0.287$ is postulated and if this material were subjected to an external load configuration equivalent to plane stress, the constitutive equation becomes:

$$\{\sigma^P\} = \begin{bmatrix} 32147.998 & 9226.476 & 0.0 \\ & 32147.998 & 0.0 \\ \text{(symmetric)} & & 11460.0 \end{bmatrix} \{\varepsilon^P\} \qquad (3a)$$

3. Elasticity in Plane Strain

In the plane strain problem, the in-plane strains are defined as:

$$\{\varepsilon^P\} = [\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}]^T .$$

Corresponding to these in-plane strains are in-plane stresses:

$$\{\sigma^P\} = [\sigma_x, \sigma_y, \tau_{xy}]^T .$$

Other stress components also exist, and these out-of-plane stresses are defined as:

$$\{\sigma^{OP}\} = [\sigma_z, \tau_{xz}, \tau_{yz}]^T ,$$

where $\{\sigma^{OP}\}$ is the vector of out-of-plane stresses.

The constitutive equation for the plane strain problem is, in terms of equation (1c):

$$\begin{Bmatrix} \varepsilon^P \\ 0 \end{Bmatrix} = \begin{bmatrix} B_{11}^* & B_{12}^* \\ B_{12}^{*\,T} & B_{22}^* \end{bmatrix} \begin{Bmatrix} \sigma^P \\ \sigma^{OP} \end{Bmatrix} \tag{4}$$

where

$$B_{11}^* = \begin{bmatrix} b_{11} & b_{12} & b_{14} \\ b_{21} & b_{22} & b_{24} \\ b_{41} & b_{42} & b_{44} \end{bmatrix} ; \quad B_{12}^* = \begin{bmatrix} b_{13} & b_{15} & b_{16} \\ b_{23} & b_{25} & b_{26} \\ b_{43} & b_{45} & b_{46} \end{bmatrix} ; \quad B_{22}^* = \begin{bmatrix} b_{33} & b_{35} & b_{36} \\ b_{53} & b_{55} & b_{56} \\ b_{63} & b_{65} & b_{66} \end{bmatrix} ;$$

and the b_{mn} are the linear coefficients in equation (1c).

From equation (4):

$$\{\sigma^{OP}\} = - [(B_{22}^*)^{-1}(B_{12}^{*\,T})]\{\sigma^P\} \tag{4a}$$

and,

$$\{\varepsilon^P\} = [B_{11}^*]\{\sigma^P\} + [B_{12}^*]\{\sigma^{OP}\}$$

$$= [B_{11}^* - (B_{12}^*)(B_{22}^*)^{-1}(B_{12}^{*\,T})]\{\sigma^P\} . \tag{4b}$$

In the plane strain problem the strain components only are in-plane: the state of stress is three-dimensional with the out-of-plane components given by equation (4a).

Alternately, equation (4b) is written:

$$\{\varepsilon^P\} = [B']\{\sigma^P\} \tag{4c}$$

where $[B']$ is a symmetric, 3×3 matrix of constants:

$$[B'] = [B_{11}^* - (B_{12}^*)(B_{22}^*)^{-1}(B_{12}^{*\,T})] ,$$

the determination of which characterizes a linear, anisotropic material in plane strain.

For perfectly isotropic materials submitted to a state of plane strain:

$$B'_{11} = \frac{1-\nu^2}{E} = B'_{22}$$

$$B'_{12} = -\frac{\nu}{1-\nu} = B'_{11}$$

$$B'_{13} = B'_{23} = 0$$

$$B'_{33} = \frac{2}{1-\nu} B'_{11}$$

where E and ν are Young's modulus and Poisson's ratio, respectively, and B'_{11}, B'_{12}, B'_{13}, B'_{22}, B'_{23}, and B'_{33} are the six elastic constants forming $[B']$.

4. Characterization of Anisotropic Materials in Planar Elasticity

For all planar problems in elasticity it is generally observed that the two-dimensional behavior of the material is defined by a constitutive law of in-plane components: a vector of in-plane components is related to a second vector of in-plane components by a symmetric, 3×3 matrix of elastic constants. The two cases of planar elasticity defined by equation (2c) and equation (4c) are not mathematically equivalent: the matrix of material constants $[A']$ is not the inverse of the matrix of material constants $[B']$. Equations (2c) and (4c) do permit, however, the writing of a general constitutive law applicable to all planar elasticity problems:

$$\{\sigma^P\} = [D]\{\varepsilon^P\} \qquad (5)$$

where $[D]$ is a 3×3, symmetric matrix equal to matrix $[A']$ for plane stress

problems and equal to $[B']^{-1}$ for plane strain problems. It is the form of equation (5) with which this study is concerned: a technique of measuring the six, independent elastic constants,

$$[D] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ \text{(symmetric)} & & C_{33} \end{bmatrix},$$

is the goal of this study in the characterization of linear, anisotropic material constants.

B. FINITE ELEMENT FORMULATION

Analysis of the basic unit of a structure is the first step in the total analysis of the structure. The basic part under consideration here is the finite element. Attention is focused on problems of planar stress, although the technique is expressed in a form generally applicable to any linear, anisotropic material. The finite elements employed are necessarily restricted to two-dimensional configurations.

1. Finite Element Geometry

A myriad of two-dimensional shapes have been used in finite element analyses [5, 21], the simplest being that of a triangle. Another very simple shape is that of a rectangle. Other shapes, from trapezoids through to curved shapes, are also available. For the purpose of the characterization of a linear, anisotropic material the simple rectangle has been selected primarily because of its inherent symmetries and ease of mathematical description.

2. Displacement Formulation for the Finite Element

The displacement formulation for a finite element is well documented [4, 5, 17, 18, 21]. Appendix A presents the details for the formulation of a 4-noded, linear element.

The elemental stiffness is given by:

$$[k]^e = \iint_s [B]^T[D][B]ds \ , \tag{6}$$

where $[k]^e$ is defined as the elemental stiffness matrix, and the indicated integration is performed across the area of the element.

The equation of static equilibrium for an element can be written as:

$$\{Q\}^e = [k]^e\{\delta\}^e \ , \tag{7}$$

where $\{Q\}^e$ is the vector of forces acting at the elemental nodes.

Appendix A provides the specific derivation of the elemental stiffness matrix $[k]^e$ for a 4-noded rectangular finite element in closed-form. The resulting symmetric, 8 × 8 matrix is shown as Figure 7 of that appendix.

In conventional finite element analysis, elemental stiffness contributions are summed at common nodal points throughout the structure by the method of direct stiffness. Forces are summed throughout the structure, and it is observed that only forces external to the structure remain because of the nullifying effects of internal nodal forces. Resultant forces are of two kinds: forces due to the reactions of imposed physical boundary conditions and forces loading the structure. The equation of equilibrium for the structure is:

$$\{R\} = [K]\{\delta\} \ , \tag{8}$$

where $\{R\}$ is the vector of resultant forces acting at each node, generally equal to zero at internal nodes of the structure; $[K]$ is the master stiffness matrix; and $\{\delta\}$ is the vector of displacements containing the displacements at each node of the structure.

Equation (6) defines elemental stiffness as a function of the particular geometry of the element and the six elastic constants. Performing the matrix multiplication indicated by equation (7) and factoring-out the vector of elastic constants, a modified elemental stiffness matrix is defined [12, 13]:

$$\{Q\}^e = [k^*]^e\{C\} \ , \tag{9}$$

where $[k^*]$ is the modified elemental stiffness matrix and $\{C\}$ is the vector of elastic constants, C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, and C_{33}. This modified elemental stiffness matrix is not square: for the 4-noded element the modified stiffness matrix is of size 8×6.

The members of the modified elemental stiffness matrix are functions of the particular geometry of the element and the nodal displacements, u_i and v_i, of the element. Figure 8, Appendix A, presents the closed-form solution for the modified stiffness matrix for the 4-noded rectangular element. Such a closed-form solution for the modified stiffness matrix is notationally straight forward, but a good deal of tedious manipulation is hidden behind the symbols: closed-form solutions have not been successfully obtained for the elemental stiffness matrix or the modified elemental stiffness matrix for an 8-noded rectangular element because of repeated algebraic difficulties.

A simplified direct summation procedure is used to form the master modified stiffness matrix, and the forces are again summed across the structure:

$$\underset{n \times 1}{\{R\}} = \underset{n \times 6}{[K^*]} \underset{6 \times 1}{\{C\}} \quad . \tag{10}$$

3. Material Characterization Using Finite Elements

Figure 8, Appendix A, presents the 8×6 modified stiffness matrix derived for the 4-noded rectangular element. A structure composed of a number of these elements will exhibit a master modified stiffness matrix of order $n \times 6$. For the planar problems of elasticity two degrees of freedom exist at each node in the structure, and n equals twice the number of joints.

The over-determined system of equations (10) can be solved as follows:

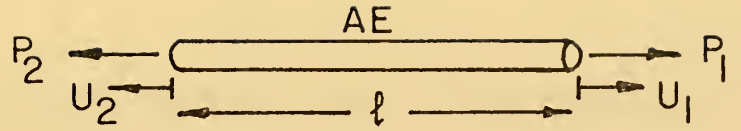
$$[K^*]^T\{R\} = [K^*]^T[K^*]\{C\}$$

$$([K^*]^T[K^*])^{-1}[K^*]^T\{R\} = \{C\} \quad , \tag{11}$$

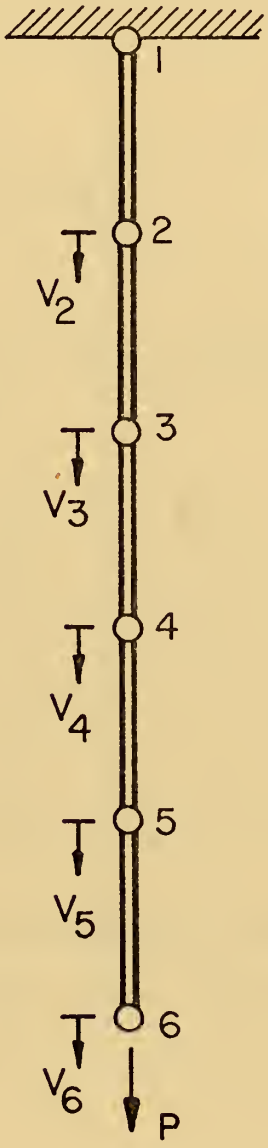
if the product $[K^*]^T[K^*]$ is non-singular.

C. CHARACTERIZATION USING FINITE ELEMENTS: AN EXAMPLE

Figure 1 presents an illustrative example of the use of finite elements in the characterization of elastic solids and is a modification after Kavanagh [12]. The problem is restricted to one-dimension and to one elastic constant, E.



TYPICAL FINITE ELEMENT, ONE-DIMENSIONAL.



STRUCTURE L= 5 ℓ
AE CONSTANT
LONGITUDINAL LOAD P.

5 FINITE ELEMENTS
6 JOINTS.

FIGURE 1 : ONE-DIMENSIONAL EXAMPLE PROBLEM ILLUSTRATING THE FINITE ELEMENT CHARACTERIZATION OF ELASTIC SOLIDS.

The example consists of a weight P, hanging from a slender rod of total length L and cross-sectional area A. The rod is divided into five finite elements, each element containing two nodes. There are a total of six structural joints, the first of which is constrained by a physically pinned connection.

The finite element in one-dimension yields an elemental stiffness matrix:

$$[k]^e = \frac{AE}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} .$$

The master stiffness matrix for the overall structure is found by the direct stiffness method:

$$[K] = \frac{AE}{\ell} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} .$$

Imposition of the boundary condition at the first joint allows eliminating the first row and the first column, and the equation for structural equilibrium is written:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P \end{Bmatrix} = \frac{AE}{\ell} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix} .$$

Performing the indicated matrix multiplication and factoring the material constant, E:

$$\underset{(5 \times 1)}{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P \end{Bmatrix}} = \underset{(5 \times 1)}{\begin{bmatrix} (2v_2 - v_3)\, A/\ell \\ (-v_2 + 2v_3 - v_4)\, A/\ell \\ (-v_3 + 2v_4 - v_5)\, A/\ell \\ (-v_4 + 2v_5 - v_6)\, A/\ell \\ (-v_5 + v_6)\, A/\ell \end{bmatrix}} \underset{(1 \times 1)}{\{E\}} .$$

Setting $A = 1.0$, $\ell = 100.0$, and $P = 10.0$, assume that the measured deflections are $v_6 = 5.0$, $v_5 = 4.0$, $v_4 = 3.0$, $v_3 = 2.0$, $v_2 = 1.0$:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10.0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.01 \end{Bmatrix} \{E\} .$$

Solving the equation for {E}:

$$[0 \quad 0 \quad 0 \quad 0 \quad 0.01] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10.0 \end{Bmatrix} = [0 \quad 0 \quad 0 \quad 0 \quad 0.01] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.01 \end{Bmatrix} \{E\}$$

$$0.10 = 0.0001 \{E\}$$

$$1000.0 = \{E\} \quad .$$

It is instructive to note that displacements calculated using this value of E would be identically those values assumed at joints 2, 3, 4, 5, and 6. In an actual experiment, however, the displacements obtained at the joints by measurement of the structure would be in error by some experimental amount. If E were known exactly, the E calculated would not be exact.

III. EXPERIMENTAL SIMULATION AND PROCEDURE

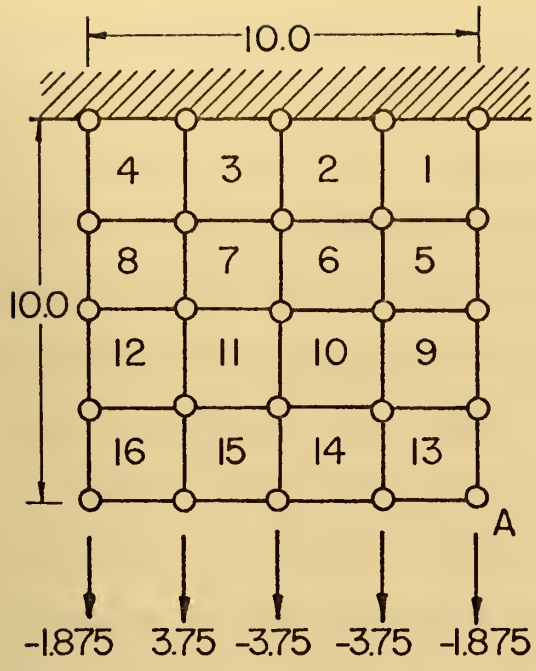
A. NODAL DISPLACEMENTS

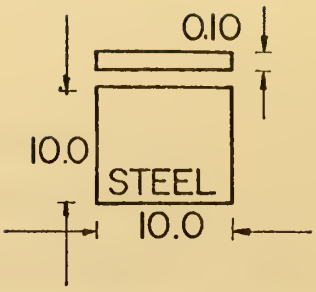
The application of the finite element method to the characterization of elastic solids is predicated on the accurate knowledge of displacements at the nodes of the structure. Small errors in obtaining displacements will cause large errors to be introduced in the values of the calculated material constants, as reported by Kavanagh [12]. It has been necessary, therefore, to employ a simulation scheme to generate the displacements associated with a given loading configuration.

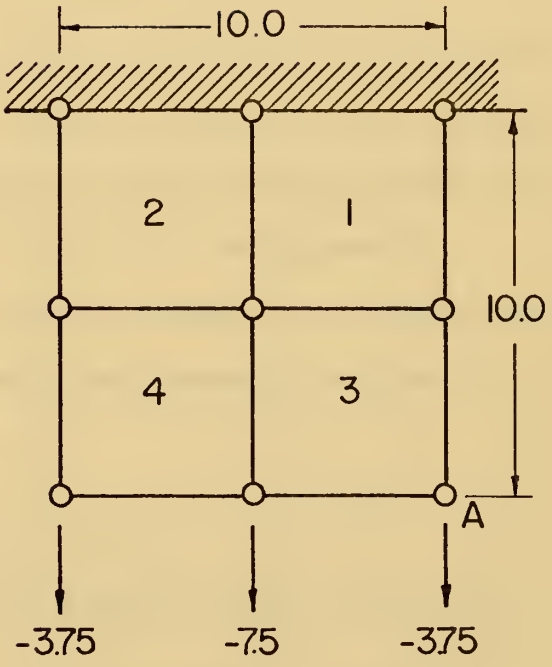
Appendix B consists of a listing of the digital computer program DPLISOP. DPLISOP is a double-precision version of PLISOP [3], a general purpose finite element program for planar problems in elasticity. DPLISOP was used to generate the nodal displacements required throughout this study.

1. Convergence Study

A convergence study using DPLISOP was performed. A steel plate, 10.0 inches square and 0.10 inches thick, was simulated in a state of plane stress. The top edge of the plate was constrained, and the bottom edge of the plate was uniformly loaded to a static load of 15.0 kips. Loading at the bottom edge nodes was simulated by the usual application of the consistent load vector concept [21]. Young's modulus and Poisson's ratio were taken to be $E = 29.50 \times 10^3$ ksi and $\nu = 0.287$, respectively. Finite element discretizations of one, four, and sixteen elements were used. Figure 2 shows these trial discretizations using the 4-noded element. Although DPLISOP is capable of utilizing 4-noded, 8-noded, and







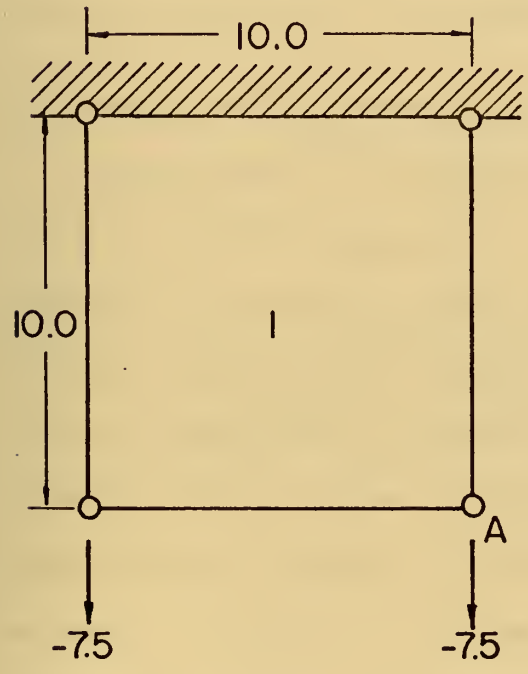


FIGURE 2: FINITE ELEMENT DISCRETIZATION FOR 4-NODED CONVERGENCE STUDY. STEEL PLATE, 10 BY 10, 0.10 THICK, TOTAL 15 KIP LOAD UNIFORMLY ALONG BOTTOM OF PLATE.

12-noded rectangular finite elements, runs were made for only the 4-noded and 8-noded elements.

Figure 3 presents the results of the convergence study for the 4-noded rectangular element, at Node A of the simulation structure. Similar results were obtained for the 8-noded rectangular element. The minimum number of elements necessary to ensure accurate displacements in both the horizontal and vertical directions was found to be four.

Implicit in the convergence study was the requirement that the experimental structure exhibit symmetries in both the horizontal and vertical directions and that elements be very nearly square to avoid geometrically induced anisotropic behavior. Figure 4 details the configuration, consisting of four, 4-noded rectangular finite elements, containing nine nodal points, and an overall 10.0 inches square geometry, selected for use in the remainder of the study.

B. COMPUTER PROGRAM DLASTIC

Program DLASTIC was written as a test program for the determination of the six elastic constants C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, and C_{33}, by direct inversion of the finite element formulation (equations (10) and (11)), in order to characterize a general, linear anisotropic material. Double-precision displacements generated in DPLISOP were used as input data to DLASTIC and the six material constants calculated. These calculated numerical values proved to be exactly the characterizing six material constants of equation (3a). These results appear as a representative output which, with the listing of DLASTIC, form Appendix C.

Element geometry used in DLASTIC is restricted to the 4-noded rectangular finite element shape.

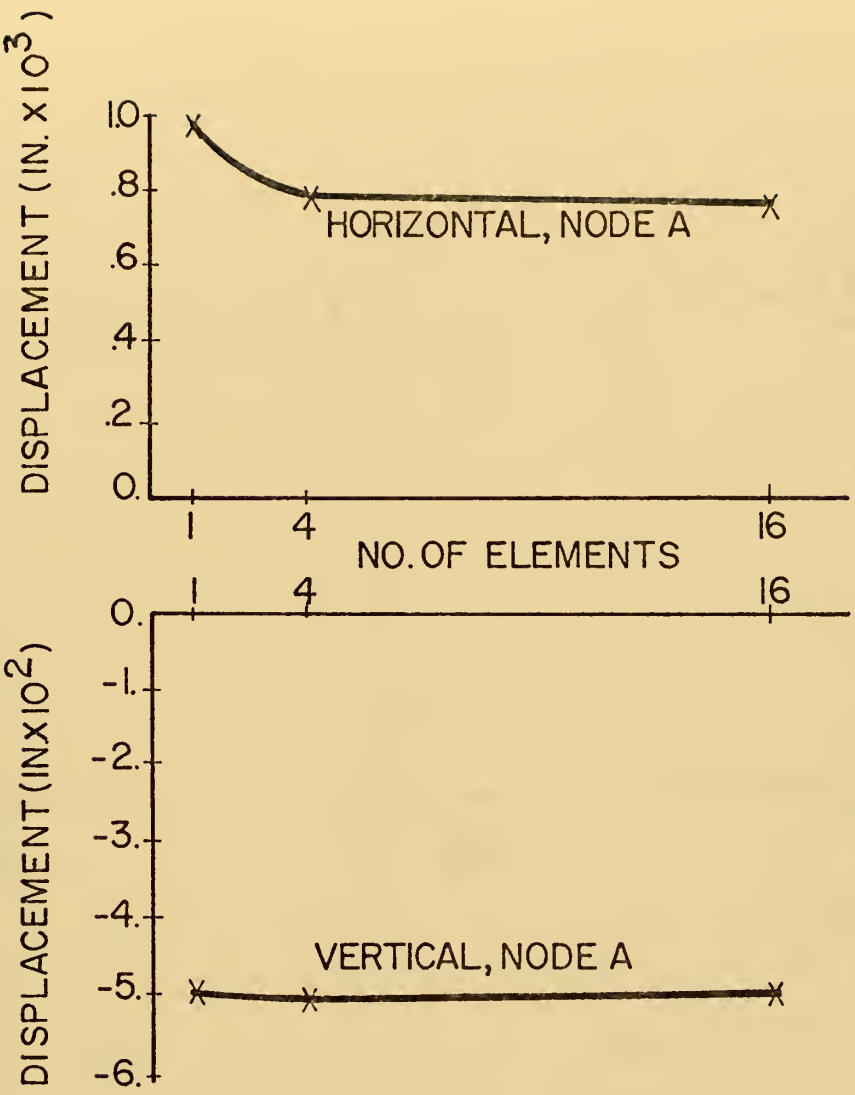


FIGURE 3: CONVERGENCE STUDY, NODE-A, USING 4-NODED RECTANGULAR FINITE ELEMENTS. STEEL PLATE IS 10 BY 10 SQUARE, 0.10 THICK, LOADED IN PLANE STRESS BY 15 KIP TOTAL LOAD UNIFORMLY DISTRIBUTED ALONG BOTTOM EDGE OF PLATE.

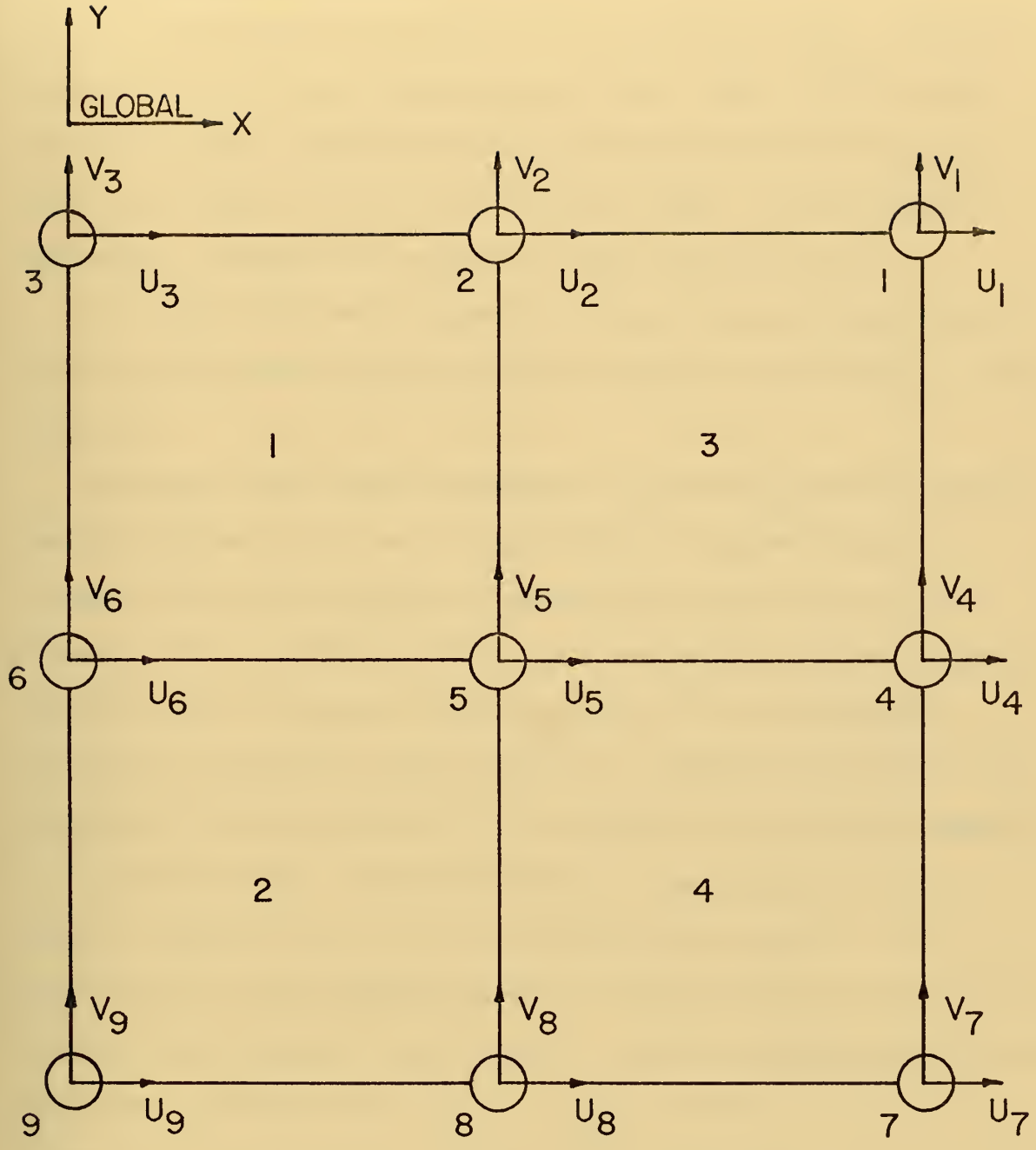


FIGURE 4 : 4-NODED 4-ELEMENT STRUCTURE SHOWING JOINT NUMBERING CONVENTION AND POSITIVE DISPLACEMENTS AT JOINTS.

1. Program Organization

Program DLASTIC was written as a modular algorithm: the MAIN program calls, in order, the subroutines INPUT, MERGE, BMULT, SINGL, INVRT, and ANSER. Subroutine MERGE, in turn, calls subroutine STIFF. Subroutine SINGL, in turn, calls subroutine EIGEN. Communication is maintained throughout the program by common blocks of storage.

The eight digit numbers appearing in the discussion which follows refer to the program sequencing numbers appearing in the program listing, Appendix C.

Subroutine INPUT (00000700-00001350) is called by the MAIN program. Punched data cards are read and echo-checked. Required input data includes structural parameter specifications, such as the number of elements and joints (NEL and NJT), nodal connectivity for every element (NCON), and the coordinates (COORD), forces (R), and displacements (U and V) for each joint in the structure. Specific directions for data preparation have been included within the MAIN program listing (Appendix C).

Subroutine MERGE (00001360-00001820) is next called by the MAIN program. MERGE forms the master modified stiffness matrix $[K^*]$, by superposition of the elemental modified stiffness matrices at appropriate common joints throughout the structure. Subroutine STIFF (00001830-00002910) is called in MERGE within a do-loop sequence indexed over the number of elements in the structure. Subroutine STIFF forms the elemental modified stiffness matrix.

Next called by the MAIN program is subroutine BMULT (00002920-00003270). In BMULT, the master modified stiffness matrix, BK, is premultiplied by

its transpose, BKT. The resulting symmetric matrix has been designated BSYM. The vector of nodal forces, R, is also premultiplied by BKT: this result has been designated BR.

The first multiplication in subroutine BMULT, the symmetric matrix BSYM, corresponds to the term $[K^*]^T[K^*]$ of equation (11). The MAIN program calls subroutine SINGL (00003280-00003530), which calls subroutine EIGEN (00003540-00005350) and a determination is made regarding the singularity or non-singularity of the matrix BSYM.

EIGEN calculates the six eigenvalues of BSYM associated with the as yet undetermined six material constants. SINGL causes the six eigenvalues to be printed as part of the program output, and inspection of this portion of the output is required by the program user to ascertain the validity of the total program output. Discussion of this requirement is deferred to the section following this discussion of program organization.

Subroutine INVRT (00005360-00006930) is next called from the MAIN program. The result of the first multiplication in subroutine BMULT, the matrix BSYM, is inverted using the Gauss-Jordan method. The determinant calculated in INVRT is ignored, for reasons which will become clear in the discussion of the eigenvalue criterion.

The vector of material constants, C, is calculated in subroutine ANSER (00006940-00007190) by premultiplying the vector BR, calculated in subroutine BMULT, by the inverted matrix BSYM, which occupies the storage location previously occupied by the non-inverted matrix. The resulting vector C, if valid, contains the six material constants C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, and C_{33}. ANSER causes the constants to be printed as program output.

DLASTIC was written in FORTRAN IV and compiled with the G-compiler under release 18 on an IBM 360/67 machine. It required 84,000 bytes of core and used approximately one-second CPU time for a typical run.

2. Eigenvalue Criterion of Matrix Singularity

A matrix is said to be singular when the determinant of the matrix is equal to zero. However, when a matrix is ill-conditioned and nearly singular the numerical evaluation of the determinant is almost invariably completely unreliable. Round-off in most algorithms is sufficient to introduce an artificial round-off value of zero that invalidates the calculation of the determinant.

The calculation of the eigenvalues of a symmetric matrix is always a well conditioned numerical problem, even when the matrix is exactly singular. A better indication of singularity and near-singularity is obtained by examining all eigenvalues appropriate to the matrix. The condition number of the matrix is then obtained as the ratio of the largest absolute value of the eigenvalues to the smallest absolute value of the eigenvalues. If this condition number is greater than 10^{12} the matrix is too nearly singular to be inverted accurately.

In program DLASTIC it was necessary to employ the eigenvalue criterion of singularity. The symmetric matrix BSYM, formed in subroutine BMULT, was found to exhibit eigenvalues in the range 10^{-6} to 10^{-9}. The condition number is formed, 10^{3}, and it is obvious that matrix BSYM is very far from any singularity.

C. DISPLACEMENT PERTURBATIONS

An exact characterization of an elastic solid was presented in a previous section (see the representative output, Appendix C). Such a

result utilized the full double-precision displacement data (sixteen digits) generated within the simulation program DPLISOP. Such exhaustive displacement data cannot be expected to be available from a physical experiment or possible test device. It was necessary, therefore, to perform a series of perturbation analyses in order to ascertain the lower limit of displacement data accuracy required for the characterization of elastic solids by the direct inversion of the finite element formulation.

A first set of truncated displacements was used to determine how sensitive the elastic constants would be to such approximations. Round-off to five, six, seven, and eight places of decimals was used (see Table 1). The elastic constants were calculated for each set of rounded displacements using program DLASTIC. The relative errors of the resulting material constants were then evaluated.

The results for the material constants and the associated errors are presented as Table 2.

Random perturbations, ranging in value from -5 to +5, were next applied to the rounded results in the fifth, sixth, seventh, and eighth places of decimals. Five different sets of displacements were obtained for each of the rounded results shown in Table 1.

Every set of displacements was used as input data for program DLASTIC, and all six material constants were obtained for each case. The results are shown in Tables 3 and 4.

JOINT	DISPLACEMENT	NON-RANDOM DISPLACEMENT PERTURBATIONS			
4 u	-0.748329139132374D-03	-0.00075	-0.000748	-0.0007483	-0.00074833
v	-0.252610022185014D-02	-0.00253	-0.002526	-0.0025261	-0.00252610
5 v	-0.235459058006389D-02	-0.00253	-0.002355	-0.0023546	-0.00235459
6 u	0.748329139132375D-03	0.00075	0.000748	0.0007483	0.00074833
v	-0.252610022185015D-02	-0.00253	-0.002526	-0.0025261	-0.00252610
7 u	-0.770070823585924D-03	-0.00077	-0.000770	-0.0007701	-0.00077007
v	-0.500613331025850D-02	-0.00501	-0.005006	-0.0050061	-0.00500613
8 v	-0.497625861993874D-02	-0.00498	-0.004976	-0.0049763	-0.00497626
9 u	0.770070823585927D-03	0.00077	0.000770	0.0007701	0.00077007
v	-0.500613331025850D-02	-0.00501	-0.005006	-0.0050061	-0.00500613
		RUN 1	RUN 2	RUN 3	RUN 4

TABLE 1 : NON-RANDOM PERTURBATIONS (SIMPLE ROUNDING) OF NON-ZERO HORIZONTAL AND VERTICAL NODAL DISPLACEMENTS, SIMULATED 10.0 BY 10.0, 0.10 THICK STEEL STRUCTURE IN PLANE STRESS. 15.0 KIPS LOAD.

EXACT VALUES, ANISOTROPIC (PL. STRESS)					
C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}
32147.998	9226.476	0.0	32147.998	0.0	11460.0

RUN 1		ERROR, %
C_{11}	32757.	1.9
C_{12}	9381.	1.7
C_{13}	0.	0.0
C_{22}	32159.	0.0
C_{23}	0.	0.0
C_{33}	11421.	0.3

RUN 2		ERROR, %
C_{11}	32093.	0.2
C_{12}	9211.	0.2
C_{13}	0.	0.0
C_{22}	32145.	0.0
C_{23}	0.	0.0
C_{33}	11471.	0.0

RUN 3		ERROR, %
C_{11}	32145.	0.0
C_{12}	9226.	0.0
C_{13}	0.	0.0
C_{22}	32148.	0.0
C_{23}	0.	0.0
C_{33}	11463.	0.0

RUN 4		ERROR, %
C_{11}	32148.	0.0
C_{12}	9226.	0.0
C_{13}	0.	0.0
C_{22}	32148.	0.0
C_{23}	0.	0.0
C_{33}	11461.	0.0

TABLE 2: THE SIX ANISOTROPIC CONSTANTS CALCULATED IN PROGRAM DLASTIC, BY ROUNDING OF DISPLACEMENT DATA.

	EXACT	1	2	3	4	5	PERTURBATION 5 Percent error				
C_{11}	32147.998	39194.	35217.	31973.	29993.	36222.	22.	9.5	0.5	6.7	13.
C_{12}	9226.476	10978.	9954.	9243.	8993.	10353.	19.	7.9	0.2	2.5	12.
C_{13}	0.0	1041.	620.	328.	-759.	-414.	--	--	--	--	--
C_{22}	32147.998	32332.	32402.	32352.	32261.	32401.	0.6	0.8	0.6	0.4	0.8
C_{23}	0.0	95.	52.	-5.	-53.	-61.	--	--	--	--	--
C_{33}	11460.761	10665.	9882.	11924.	11314.	8591.	6.9	14.	4.0	1.3	25.
							PERTURBATION 6 Percent error				
C_{11}	32147.998	31290.	31660.	31553.	32201.	32136.	2.7	1.5	1.9	0.2	0.0
C_{12}	9226.476	9005.	9108.	9058.	9237.	9214.	2.4	1.3	1.8	0.1	0.1
C_{13}	0.0	32.	-105.	75.	-13.	-13.	--	--	--	--	--
C_{22}	32147.998	32108.	32115.	32130.	32174.	32136.	0.1	0.1	0.0	0.0	0.0
C_{23}	0.0	13.	-18.	6.	-7.	4.	--	--	--	--	--
C_{33}	11460.761	11594.	11518.	11699.	11481.	11618.	1.2	0.5	2.1	0.2	1.4

TABLE 3: THE SIX ANISOTROPIC MATERIAL CONSTANTS CALCULATED IN PROGRAM DLASTIC, BASED ON RANDOM PERTURBATIONS TO THE FIFTH AND SIXTH DECIMAL PLACES OF DISPLACEMENTS.

	EXACT	1	2	3	4	5	PERTURBATION 7 Percent error				
C_{11}	32147.998	32153.	32102.	32074.	32094.	32103.	0.0	0.1	0.2	0.2	0.1
C_{12}	9226.476	9227.	9215.	9211.	9215.	9214.	0.0	0.1	0.2	0.1	0.1
C_{13}	0.0	-6.	-3.	-9.	-14.	-4.	--	--	--	--	--
C_{22}	32147.998	32146.	32144.	32145.	32145.	32143.	0.0	0.0	0.0	0.0	0.0
C_{23}	0.0	-2.	-2.	-1.	-2.	-1.	--	--	--	--	--
C_{33}	11460.761	11447.	11467.	11465.	11458.	11468.	0.1	0.0	0.0	0.0	0.0

							PERTURBATION 8 Percent error				
C_{11}	32147.998	32142.	32154.	32145.	32155.	32141.	0.0	0.0	0.0	0.0	0.0
C_{12}	9226.476	9225.	9228.	9226.	9228.	9225.	0.0	0.0	0.0	0.0	0.0
C_{13}	0.0	0.	0.	0.	-1.	-1.	0.0	0.0	0.0	--	--
C_{22}	32147.998	32148.	32148.	32148.	32148.	32147.	0.0	0.0	0.0	0.0	0.0
C_{23}	0.0	0.	0.	0.	0.	0.	0.0	0.0	0.0	0.0	0.0
C_{33}	11460.761	11463.	11459.	11459.	11462.	11463.	0.0	0.0	0.0	0.0	0.0

TABLE 4: THE SIX ANISOTROPIC MATERIAL CONSTANTS CALCULATED IN PROGRAM DLASTIC, BASED ON RANDOM PERTURBATIONS TO THE SEVENTH AND EIGHTH DECIMAL PLACES OF DISPLACEMENTS.

IV. RESULTS AND CONCLUSIONS

The six, anisotropic material constants appropriate to the plane stress elasticity problem and the particular material were calculated in equation (3a). Using the full double-precision displacement data generated by simulating an elastic structure in a state of plane stress program DLASTIC was used to calculate numerical values for these same six, anisotropic constants. Both sets of numerical constants were identical. For the purpose of discussion these values for the six material constants have been considered the exact numerical values of the constants.

Program DLASTIC required closed-form solutions to the stiffness matrices of the finite element formulation. Closed-form solutions for the elemental stiffness matrix, $[k]$, and the modified elemental stiffness matrix, $[k^*]$, were successfully obtained for the 4-noded, rectangular finite element. These results form Figures 7 and 8, Appendix A.

A. PERTURBATION RESULTS

Reasonable numerical values for the six elastic constants were obtained, based on approximate displacement data generated by rounding to as few as five decimal places. Table 2 presents the calculated values for these constants. Eigenvalues were also calculated: the four runs using simple roundings as approximations returned eigenvalues in the range 10^{-6} to 10^{-9}. Approximate displacement data to the fifth decimal place caused the returned numerical values of the constants to be within two percent of the exact numerical values, and approximate

displacement data to the sixth, seventh, and eighth decimal places resulted in numerical values within half-of-one percent of the exact values of the material constants.

Twenty sets of displacement data were generated for the random analyses. All twenty data sets were run in program DLASTIC and numerical values for the elastic constants calculated.

The numerical values for the calculated elastic constants are presented in Tables 3 and 4. The calculated eigenvalues for all random perturbations were within the range of 10^{-6} to 10^{-9}. Comparison of these calculated material constants to the exact values gives an idea of the sensitivity of the elastic constants to the accuracy of the displacement data.

Random perturbations applied to the results rounded to five decimal places gave the largest errors in the elastic constants. Worse values were consistently the calculated constant C_{13}, the values of which ranged from -759.5 ksi to 1041.0 ksi.

Simulated structural displacements accurate to 10^{-5} inches, the sixth decimal place of displacements being random, returned numerical values for the material constants generally less than three percent in error. Values of C_{13}, though not identically zero, were sufficiently small to consider them calculated zero except in two cases. Larger errors were experienced in the calculated values for C_{11} and C_{12} for all five sets.

Actual displacement measurements for a test device or physical experiment could be expected to require a horizontal displacement data precision of one part in eight hundred (1/800) and a vertical

displacement precision of one part in five hundred (1/500). These precisions could be expected to yield numerical values for the material constants in the range of two to three percent error.

Perturbations to the seventh and eighth decimal places returned material constants less than one percent in error (Table 4). An actual test device would require a horizontal measurement precision of one part in eight thousand (1/8000) and a vertical precision of one part in five thousand (1/5000), based on the simulated structural displacements being accurate to 10^{-6} inches and the seventh decimal place of displacements being random. The test device could be expected to yield numerical values for the material constants much less than one percent in error.

The numerical values returned from perturbations of displacement data to the eighth decimal place indicated required precisions of one part in eighty thousand (1/80000), horizontal, and one part in fifty thousand (1/50000), vertical. These precisions could be expected to yield numerical values for the material constants much less than one percent in error.

B. CONCLUSIONS

As a consequence of this study of the finite element technique, as utilized in the characterization of linear, anisotropic elastic solids, several conclusions can be drawn:

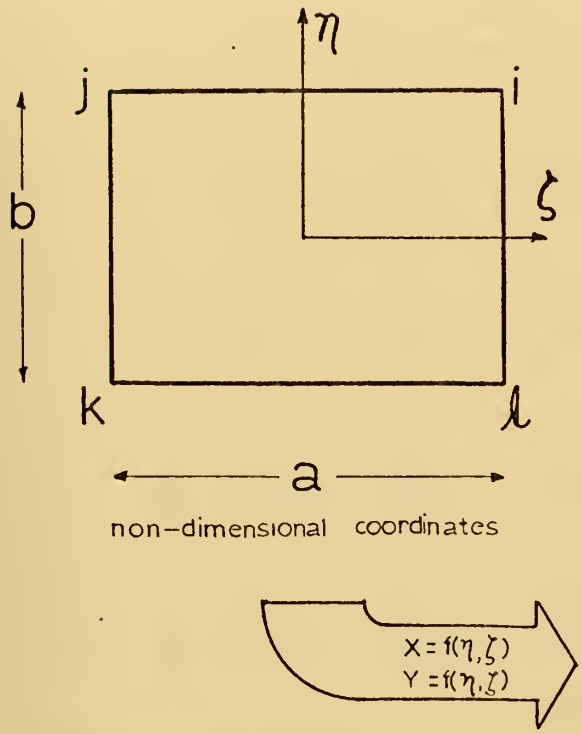
1. The finite element formulation under study can be applied to the mechanical characterization of linear, anisotropic solids.
2. The accuracy of characterization is dependent on the sensitivity of the finite element formulation.
3. The sensitivity of the finite element formulation is dependent on the precision with which displacement data is measured.

4. A device designed to determine the six linear, anisotropic elastic constants to an accuracy of three percent will require a precision in displacement measurements of at least one part in eight hundred (1/800).

5. A device designed to determine the six linear, anisotropic elastic constants to an accuracy of one percent will require a precision in displacement measurements of at least one part in eight thousand (1/8000).

APPENDIX A

Derivation of stiffness matrices for a 4-noded finite element



$$N_r = \frac{1}{4}(1+\zeta_o)(1+\eta_o)$$

$$\zeta_o = \zeta_r \zeta \qquad \zeta_r = \pm 1$$

$$\eta_o = \eta_r \eta \qquad \eta_r = \pm 1$$

$$N_i = \frac{1}{4}(1+\zeta)(1+\eta)$$
$$N_j = \frac{1}{4}(1-\zeta)(1+\eta)$$
$$N_k = \frac{1}{4}(1-\zeta)(1-\eta)$$
$$N_l = \frac{1}{4}(1+\zeta)(1-\eta)$$

$$x = N_i x_i + N_j x_j + N_k x_k + N_l x_l$$
$$y = N_i y_i + N_j y_j + N_k y_k + N_l y_l$$

$$\frac{\partial N_i}{\partial \zeta} = \frac{1}{4}(1+\eta) \quad \frac{\partial N_i}{\partial \eta} = \frac{1}{4}(1+\zeta) \quad \frac{\partial N_j}{\partial \zeta} = -\frac{1}{4}(1+\eta) \quad \frac{\partial N_j}{\partial \eta} = \frac{1}{4}(1-\zeta)$$

$$\frac{\partial N_k}{\partial \zeta} = \frac{1}{4}(\eta-1) \quad \frac{\partial N_k}{\partial \eta} = \frac{1}{4}(\zeta-1) \quad \frac{\partial N_\ell}{\partial \zeta} = \frac{1}{4}(1-\eta) \quad \frac{\partial N_\ell}{\partial \eta} = -\frac{1}{4}(1+\zeta)$$

$$\frac{\partial x}{\partial \zeta} = \frac{\partial N_i}{\partial \zeta} x_i + \frac{\partial N_j}{\partial \zeta} x_j + \frac{\partial N_k}{\partial \zeta} x_k + \frac{\partial N_\ell}{\partial \zeta} x_\ell$$

$$\frac{\partial x}{\partial \eta} = \frac{\partial N_i}{\partial \eta} x_i + \frac{\partial N_j}{\partial \eta} x_j + \frac{\partial N_k}{\partial \eta} x_k + \frac{\partial N_\ell}{\partial \eta} x_\ell$$

$$\frac{\partial y}{\partial \zeta} = \frac{\partial N_i}{\partial \zeta} y_i + \frac{\partial N_j}{\partial \zeta} y_j + \frac{\partial N_k}{\partial \zeta} y_k + \frac{\partial N_\ell}{\partial \zeta} y_\ell$$

$$\frac{\partial y}{\partial \eta} = \frac{\partial N_i}{\partial \eta} y_i + \frac{\partial N_j}{\partial \eta} y_j + \frac{\partial N_k}{\partial \eta} y_k + \frac{\partial N_\ell}{\partial \eta} y_\ell$$

Define the Jacobian, [J]:

$$[J(\eta,\zeta)] = \begin{bmatrix} \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_i}{\partial \zeta} & \frac{\partial N_j}{\partial \zeta} & \frac{\partial N_k}{\partial \zeta} & \frac{\partial N_\ell}{\partial \zeta} \\ \frac{\partial N_i}{\partial \eta} & \frac{\partial N_j}{\partial \eta} & \frac{\partial N_k}{\partial \eta} & \frac{\partial N_\ell}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_i & y_i \\ x_k & y_j \\ x_k & y_k \\ x_\ell & y_\ell \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \qquad \text{(a)}$$

Define the displacement field, $\{\delta\} = \begin{Bmatrix} u \\ v \end{Bmatrix}$

$$u = N_i u_i + N_j u_j + N_k u_k + N_\ell u_\ell$$

$$v = N_i v_i + N_j v_j + N_k v_k + N_\ell v_\ell \qquad \text{(b)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial N_i}{\partial x} u_i + \frac{\partial N_j}{\partial x} u_j + \frac{\partial N_k}{\partial x} u_k + \frac{\partial N_\ell}{\partial x} u_\ell$$

$$\frac{\partial u}{\partial y} = \frac{\partial N_i}{\partial y} u_i + \frac{\partial N_j}{\partial y} u_j + \frac{\partial N_k}{\partial y} u_k + \frac{\partial N_\ell}{\partial y} u_\ell$$

$$\frac{\partial v}{\partial x} = \frac{\partial N_i}{\partial x} v_i + \frac{\partial N_j}{\partial x} v_j + \frac{\partial N_k}{\partial x} v_k + \frac{\partial N_\ell}{\partial x} v_\ell$$

$$\frac{\partial v}{\partial y} = \frac{\partial N_i}{\partial y} v_i + \frac{\partial N_j}{\partial y} v_j + \frac{\partial N_k}{\partial y} v_k + \frac{\partial N_\ell}{\partial y} v_\ell$$

In general,

$$\frac{\partial N_r}{\partial x} = \frac{\partial N_r}{\partial \zeta} \frac{\partial \zeta}{\partial x} + \frac{\partial N_r}{\partial \eta} \frac{\partial \eta}{\partial x}$$

$$\frac{\partial N_r}{\partial y} = \frac{\partial N_r}{\partial \zeta} \frac{\partial \zeta}{\partial y} + \frac{\partial N_r}{\partial \eta} \frac{\partial \eta}{\partial y}$$

or,

$$\frac{\partial N_r}{\partial \zeta} = \frac{\partial N_r}{\partial x} \frac{\partial x}{\partial \zeta} + \frac{\partial N_r}{\partial y} \frac{\partial y}{\partial \zeta} \qquad \text{(c)}$$

$$\frac{\partial N_r}{\partial \eta} = \frac{\partial N_r}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial N_r}{\partial y} \frac{\partial y}{\partial \eta}$$

Thus,

$$\begin{Bmatrix} \frac{\partial N_r}{\partial \zeta} \\ \\ \frac{\partial N_r}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} \\ \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_r}{\partial x} \\ \\ \frac{\partial N_r}{\partial y} \end{Bmatrix}$$

$$= [J(\eta,\zeta)] \begin{Bmatrix} \frac{\partial N_r}{\partial x} \\ \\ \frac{\partial N_r}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial N_r}{\partial x} \\ \\ \frac{\partial N_r}{\partial y} \end{Bmatrix} = [J(\eta,\zeta)]^{-1} \begin{Bmatrix} \frac{\partial N_r}{\partial \zeta} \\ \\ \frac{\partial N_r}{\partial \eta} \end{Bmatrix} \qquad \text{(d)}$$

where, $$[J(\eta,\zeta)]^{-1} = 2 \begin{bmatrix} 1/A & 0 \\ \\ 0 & 1/B \end{bmatrix} .$$

For each r, $r = (i, j, k, \ell)$:

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \frac{1+\eta}{A} \\ \\ \frac{1+\zeta}{B} \end{Bmatrix} \qquad \begin{Bmatrix} \frac{\partial N_j}{\partial x} \\ \\ \frac{\partial N_j}{\partial y} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \frac{-(1+\eta)}{A} \\ \\ \frac{(1-\zeta)}{B} \end{Bmatrix} \qquad \text{(e)}$$

$$\begin{Bmatrix} \frac{\partial N_k}{\partial x} \\ \\ \frac{\partial N_k}{\partial y} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \frac{(\eta-1)}{A} \\ \\ \frac{(\zeta-1)}{B} \end{Bmatrix} \qquad \begin{Bmatrix} \frac{\partial N_\ell}{\partial x} \\ \\ \frac{\partial N_\ell}{\partial y} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \frac{1-\eta}{A} \\ \\ \frac{-(1+\zeta)}{B} \end{Bmatrix} \qquad \text{(e)}$$

In general, for any two-dimensional element:

$$[B_r] = \begin{bmatrix} \frac{\partial N_r}{\partial x} & 0 \\ \\ 0 & \frac{\partial N_r}{\partial y} \\ \\ \frac{\partial N_r}{\partial y} & \frac{\partial N_r}{\partial x} \end{bmatrix} \qquad \text{(f)}$$

and $[B] = [B_i \quad B_j \quad B_k \quad B_\ell]$ for the 4-noded rectangular element.

Specifically:

$$[B] = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_j}{\partial x} & 0 & \frac{\partial N_k}{\partial x} & 0 & \frac{\partial N_\ell}{\partial x} & 0 \\ \\ 0 & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_j}{\partial y} & 0 & \frac{\partial N_k}{\partial y} & 0 & \frac{\partial N_\ell}{\partial y} \\ \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial y} & \frac{\partial N_k}{\partial x} & \frac{\partial N_\ell}{\partial y} & \frac{\partial N_\ell}{\partial x} \end{bmatrix} \qquad \text{(g)}$$

Substituting the functions from equation (e) into equation (g), the matrix of first partial derivatives of shape functions for a 4-noded rectangular finite element, [B], is formed, (Figure 5).

The matrix of elastic constants, for planar elasticity is:

$$[D] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ \text{(symmetric)} & & C_{33} \end{bmatrix} \qquad \text{(h)}$$

where C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, and C_{33} are the six elastic constants.

The products

$$[D][B], \quad \text{and}$$

$$[B]^T[D][B]$$

are then formed (Figure 6), where

$$d^2 = \frac{(1+\eta)^2}{A^2} \qquad h^2 = \frac{(1-\zeta)^2}{B^2} \qquad gf = \frac{(1-\eta)(1+\zeta)}{AB}$$

$$g^2 = \frac{(1+\zeta)^2}{B^2} \qquad df = \frac{(1-\eta)^2}{A^2} \qquad gh = \frac{(1-\zeta^2)}{B^2}$$

$$f^2 = \frac{(1-\eta)^2}{A^2} \qquad dg = \frac{(1+\eta)(1+\zeta)}{AB} \qquad fh = \frac{(1-\eta)(1-\zeta)}{AB} \qquad \text{(i)}$$

$$dh = \frac{(1+\eta)(1-\zeta)}{AB}$$

$$[B] = \frac{1}{2}\left[\begin{array}{c|c|c|c|c|c|c|c} \frac{1}{A}(1+\eta) & 0 & -\frac{1}{A}(1-\eta) & 0 & \frac{1}{A}(\eta-1) & 0 & \frac{1}{A}(1-\eta) & 0 \\ \hline 0 & \frac{1}{B}(1+\zeta) & 0 & \frac{1}{B}(1-\zeta) & 0 & \frac{1}{B}(\zeta-1) & 0 & -\frac{1}{B}(1+\zeta) \\ \hline \frac{1}{B}(1+\zeta) & \frac{1}{A}(1+\eta) & \frac{1}{B}(1-\zeta) & -\frac{1}{A}(1+\eta) & \frac{1}{B}(\zeta-1) & \frac{1}{A}(\eta-1) & -\frac{1}{B}(1+\zeta) & \frac{1}{A}(1-\eta) \end{array}\right]$$

LET, $d = \frac{1}{A}(1+\eta)$

$f = \frac{1}{A}(1-\eta)$

$g = \frac{1}{B}(1+\zeta)$

$h = \frac{1}{B}(1-\zeta)$

THEN,

$$[B] = \frac{1}{2}\begin{bmatrix} d & 0 & -d & 0 & -f & 0 & f & 0 \\ 0 & g & 0 & h & 0 & -h & 0 & -g \\ g & d & h & -d & -h & -f & -g & f \end{bmatrix}$$

FIGURE 5: [B], THE MATRIX OF FIRST-PARTIAL DERIVATIVES OF SHAPE FUNCTIONS FOR A 4-NODED RECTANGULAR FINITE ELEMENT.

$$[D]\,[B] = \frac{1}{2}\begin{bmatrix} (c_{11}d + c_{13}g) & (c_{12}g + c_{13}d) & (-c_{11}d + c_{13}h) & (c_{12}h - c_{13}d) & (-c_{11}f - c_{13}h) & (-c_{12}h - c_{13}f) & (c_{11}f - c_{13}g) & (-c_{12}g + c_{13}f) \\ (c_{12}d + c_{23}g) & (c_{22}g + c_{23}d) & (-c_{12}d + c_{23}h) & (c_{22}h - c_{23}d) & (-c_{12}f - c_{23}h) & (-c_{22}h - c_{23}f) & (c_{12}f - c_{23}g) & (-c_{22}g + c_{23}f) \\ (c_{13}d + c_{33}g) & (c_{23}g + c_{33}d) & (-c_{13}d + c_{33}h) & (c_{23}h - c_{33}d) & (-c_{13}f - c_{33}h) & (-c_{23}h - c_{33}f) & (c_{13}f - c_{33}g) & (-c_{23}g + c_{33}f) \end{bmatrix}$$

AND,

$$[B]^T [D]\,[B] = \frac{1}{4}\begin{bmatrix} (c_{11}d^2 + c_{13}gd + c_{13}gd + c_{33}g^2) & (c_{12}gd + c_{13}d^2 + c_{23}g + c_{33}gd) & (-c_{11}d^2 + c_{13}hd - c_{13}gd + c_{33}hg) & (c_{12}hd - c_{13}d^2 + c_{23}hg - c_{33}dg) & (-c_{11}fd - c_{13}hd - c_{13}gf - c_{33}hg) & (-c_{12}hd - c_{13}fd - c_{23}gh - c_{33}gf) & (c_{11}fd - c_{13}gd + c_{23}g + c_{33}fg) & (-c_{12}gd + c_{13}fd - c_{23}g + c_{33}fg) \\ & (c_{22}g^2 + c_{23}dg + c_{23}gd + c_{33}d^2) & (-c_{12}gd + c_{23}hg - c_{13}d + c_{33}hd) & (c_{22}hg - c_{23}dg + c_{23}hd - c_{33}d^2) & (-c_{12}fg - c_{23}gh - c_{13}fd - c_{33}hd) & (-c_{22}hg - c_{23}fg - c_{23}hd - c_{33}fd) & (c_{12}fg - c_{23}g^2 + c_{13}fd + c_{33}gd) & (-c_{22}g^2 + c_{23}gf - c_{23}gd + c_{33}fd) \\ & & (c_{11}d^2 - c_{13}hd - c_{13}dh + c_{33}h^2) & (-c_{12}hd + c_{13}d^2 + c_{23}h - c_{33}dh) & (c_{11}fd + c_{13}hd - c_{13}fh - c_{33}h^2) & (c_{12}hd + c_{13}fd - c_{23}h^2 - c_{33}fh) & (-c_{11}fd + c_{13}gd + c_{13}fh - c_{33}gh) & (c_{12}gd - c_{13}fd - c_{23}gh + c_{33}fh) \\ & & & (c_{22}h^2 - c_{23}dh - c_{23}dh + c_{33}d^2) & (-c_{12}fh - c_{23}h^2 + c_{13}fd + c_{33}hd) & (-c_{22}h^2 - c_{23}fh + c_{23}hd + c_{33}fd) & (c_{12}fh - c_{23}gh - c_{13}fd + c_{33}gd) & (-c_{22}gh + c_{23}fh + c_{23}gd - c_{33}fd) \\ & & & & (c_{11}f^2 + c_{13}hf + c_{13}fh + c_{33}h^2) & (c_{12}hf + c_{13}f^2 + c_{23}h^2 + c_{33}fh) & (-c_{11}f^2 + c_{13}gf - c_{13}fh + c_{33}gh) & (c_{12}gf - c_{13}f^2 + c_{23}gh - c_{33}fh) \\ & & & & & (c_{22}h + c_{23}fh + c_{23}hf + c_{33}f^2) & (-c_{12}fh + c_{23}gh - c_{13}f^2 + c_{33}gf) & (c_{22}gh - c_{23}fh + c_{23}gf - c_{33}f^2) \\ & & & & & & (c_{11}f^2 - c_{13}gf - c_{13}fg + c_{33}g^2) & (-c_{12}gf + c_{13}f^2 + c_{23}g^2 - c_{33}fg) \\ & & & & & & & (c_{22}g^2 - c_{23}fg - c_{23}gf + c_{33}f^2) \end{bmatrix}$$

(SYMMETRIC)

FIGURE 6

The elemental stiffness is defined:

$$[k]^e = \int_y \int_x [B]^T[D][B]dxdy \ . \qquad (j)$$

The product $[B]^T[D][B]$, Figure 6, and equation (i) are functions in (η,ζ). With $dxdy = \det [J(\eta,\zeta)]$:

$$[k]^e = \int_{-1}^{1} \int_{-1}^{1} [B]^T[D][B] \det [J(\eta,\zeta)]d\eta d\zeta \qquad (k)$$

where $\det [J(\eta,\zeta)] = \frac{1}{4} AB$ and the indicated integration is performed in the non-dimensionalized coordinate system. Performing this integration, the elemental stiffness matrix results (Figure 7).

The equation of static equilibrium for the element is written:

$$\{Q\}^e = [k]^e\{\delta\}^e \qquad (\ell)$$

where $\{Q\}^e$ is the vector of forces acting at the nodes of the element; $[k]^e$ is the elemental stiffness matrix; and $\{\delta\}^e$ is the vector of elemental displacements,

$$\{\delta\} = \begin{Bmatrix} \delta_i \\ \delta_j \\ \delta_k \\ \delta_\ell \end{Bmatrix} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \\ u_\ell \\ v_\ell \end{Bmatrix} \qquad (m)$$

$$
\begin{bmatrix}
\frac{1}{3}c_{11}\frac{B}{A}+\frac{1}{2}c_{13}+\frac{1}{3}c_{33}\frac{A}{B} & \frac{1}{3}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}+\frac{1}{4}c_{33}+\frac{1}{3}c_{23}\frac{A}{B} & -\frac{1}{3}c_{11}\frac{B}{A}+\frac{1}{6}c_{33}\frac{A}{B} & -\frac{1}{3}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}-\frac{1}{4}c_{33}+\frac{1}{6}c_{23}\frac{A}{B} & -\frac{1}{6}c_{11}\frac{B}{A}-\frac{1}{2}c_{13}-\frac{1}{6}c_{33}\frac{A}{B} & -\frac{1}{6}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}-\frac{1}{4}c_{33}-\frac{1}{6}c_{23}\frac{A}{B} & \frac{1}{6}c_{11}\frac{B}{A}-\frac{1}{3}c_{33}\frac{A}{B} & \frac{1}{6}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}+\frac{1}{4}c_{33}-\frac{1}{3}c_{23}\frac{A}{B} \\
 & \frac{1}{3}c_{33}\frac{B}{A}+\frac{1}{2}c_{23}+\frac{1}{3}c_{22}\frac{A}{B} & -\frac{1}{3}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}+\frac{1}{4}c_{33}+\frac{1}{6}c_{23}\frac{A}{B} & -\frac{1}{3}c_{33}\frac{B}{A}+\frac{1}{6}c_{22}\frac{A}{B} & -\frac{1}{6}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}-\frac{1}{4}c_{33}-\frac{1}{6}c_{23}\frac{A}{B} & -\frac{1}{6}c_{33}\frac{B}{A}-\frac{1}{2}c_{23}-\frac{1}{6}c_{22}\frac{A}{B} & \frac{1}{6}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}-\frac{1}{4}c_{33}-\frac{1}{3}c_{23}\frac{A}{B} & \frac{1}{6}c_{33}\frac{B}{A}-\frac{1}{3}c_{22}\frac{A}{B} \\
 & & \frac{1}{3}c_{11}\frac{B}{A}-\frac{1}{2}c_{13}+\frac{1}{3}c_{33}\frac{A}{B} & \frac{1}{3}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}-\frac{1}{4}c_{33}+\frac{1}{3}c_{23}\frac{A}{B} & \frac{1}{6}c_{11}\frac{B}{A}-\frac{1}{3}c_{33}\frac{A}{B} & \frac{1}{6}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}-\frac{1}{4}c_{33}-\frac{1}{3}c_{23}\frac{A}{B} & -\frac{1}{6}c_{11}\frac{B}{A}+\frac{1}{2}c_{13}-\frac{1}{6}c_{33}\frac{A}{B} & -\frac{1}{6}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}+\frac{1}{4}c_{33}-\frac{1}{6}c_{23}\frac{A}{B} \\
 & & & \frac{1}{3}c_{33}\frac{B}{A}-\frac{1}{2}c_{23}+\frac{1}{3}c_{22}\frac{A}{B} & \frac{1}{6}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}+\frac{1}{4}c_{33}-\frac{1}{3}c_{23}\frac{A}{B} & \frac{1}{6}c_{33}\frac{B}{A}-\frac{1}{3}c_{22}\frac{A}{B} & -\frac{1}{6}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}+\frac{1}{4}c_{33}-\frac{1}{6}c_{23}\frac{A}{B} & -\frac{1}{6}c_{33}\frac{B}{A}+\frac{1}{2}c_{23}-\frac{1}{6}c_{22}\frac{A}{B} \\
 & & & & \frac{1}{3}c_{11}\frac{B}{A}+\frac{1}{2}c_{13}+\frac{1}{3}c_{33}\frac{A}{B} & \frac{1}{3}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}+\frac{1}{4}c_{33}+\frac{1}{3}c_{23}\frac{A}{B} & -\frac{1}{3}c_{11}\frac{B}{A}+\frac{1}{6}c_{33}\frac{A}{B} & -\frac{1}{3}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}-\frac{1}{4}c_{33}+\frac{1}{6}c_{23}\frac{A}{B} \\
 & & & & & \frac{1}{3}c_{33}\frac{B}{A}+\frac{1}{2}c_{23}+\frac{1}{3}c_{22}\frac{A}{B} & -\frac{1}{3}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}+\frac{1}{4}c_{33}+\frac{1}{6}c_{23}\frac{A}{B} & -\frac{1}{3}c_{33}\frac{B}{A}+\frac{1}{6}c_{22}\frac{A}{B} \\
 & & & & & & \frac{1}{3}c_{11}\frac{B}{A}-\frac{1}{2}c_{13}+\frac{1}{3}c_{33}\frac{A}{B} & \frac{1}{3}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}-\frac{1}{4}c_{33}+\frac{1}{3}c_{23}\frac{A}{B} \\
 & & & & & & & \frac{1}{3}c_{33}\frac{B}{A}-\frac{1}{2}c_{23}+\frac{1}{3}c_{22}\frac{A}{B}
\end{bmatrix}
$$

(SYMMETRIC)

FIGURE 7: ELEMENTAL STIFFNESS MATRIX $[k]^e$, FOR A 4-NODED RECTANGULAR FINITE ELEMENT. CLOSED-FORM SOLUTION IN TERMS OF ELEMENT DIMENSIONS (A,B) AND THE VECTOR OF LINEAR, ANISOTROPIC MATERIAL CONSTANTS ($C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}$) MATRIX $[k]^e$ IS SYMMETRIC.

Performing the multiplication indicated by equation (ℓ) and factoring the six material constants:

$$\{Q\}^e = [k^*]^e\{C\} \qquad (n)$$

where $[k^*]^e$ is the modified elemental stiffness matrix, in terms of (m), (see Figure 8), and $\{C\}$ is the vector of material constants of equation (h).

$$
\begin{bmatrix}
\frac{1}{3}\frac{B}{A}(u_l-u_j-\frac{1}{2}u_k+\frac{1}{2}u_l) & \frac{1}{4}(v_i+v_j-v_k-v_l) & \frac{1}{2}(u_i-u_k)+\frac{1}{3}\frac{B}{A}(v_i-v_j-\frac{1}{2}v_k+\frac{1}{2}v_l) & 0 & \frac{1}{3}\frac{A}{B}(v_i+\frac{1}{2}v_j-\frac{1}{2}v_k-v_l) & \frac{1}{4}(v_i-v_j-v_k+v_l)+\frac{1}{3}\frac{A}{B}(u_i+\frac{1}{2}u_j-\frac{1}{2}u_k-u_l) \\
0 & \frac{1}{4}(u_i-u_j-u_k+u_l) & \frac{1}{3}\frac{B}{A}(u_i-u_j-\frac{1}{2}u_k+\frac{1}{2}u_l) & \frac{1}{3}\frac{A}{B}(v_i+\frac{1}{2}v_j-\frac{1}{2}v_k-v_l) & \frac{1}{2}(v_i-v_k)+\frac{1}{3}\frac{A}{B}(u_i+\frac{1}{2}u_j-\frac{1}{2}u_k-u_l) & \frac{1}{4}(u_i+u_j-u_k-u_l)+\frac{1}{3}\frac{B}{A}(v_i-v_j-\frac{1}{2}v_k+\frac{1}{2}v_l) \\
\frac{1}{3}\frac{B}{A}(-u_i+u_j+\frac{1}{2}u_k-\frac{1}{2}u_l) & \frac{1}{4}(-v_i-v_j+v_k+v_l) & \frac{1}{2}(-u_j+u_l)+\frac{1}{3}\frac{B}{A}(-v_i+v_j+\frac{1}{2}v_k-\frac{1}{2}v_l) & 0 & \frac{1}{3}\frac{A}{B}(\frac{1}{2}v_i+v_j-v_k-\frac{1}{2}v_l) & \frac{1}{4}(v_i-v_j-v_k+v_l)+\frac{1}{3}\frac{A}{B}(\frac{1}{2}u_i+u_j-u_k-\frac{1}{2}u_l) \\
0 & \frac{1}{4}(u_i-u_j-u_k+u_l) & \frac{1}{3}\frac{B}{A}(-u_i+u_j+\frac{1}{2}u_k-\frac{1}{2}u_l) & \frac{1}{3}\frac{A}{B}(\frac{1}{2}v_i-v_j-v_k-\frac{1}{2}v_l) & \frac{1}{2}(-v_j+v_l)+\frac{1}{3}\frac{A}{B}(\frac{1}{2}u_i+u_j-u_k-\frac{1}{2}u_l) & \frac{1}{4}(-u_i-u_j+u_k+u_l)+\frac{1}{3}\frac{B}{A}(-v_i+v_j+\frac{1}{2}v_k-\frac{1}{2}v_l) \\
\frac{1}{3}\frac{B}{A}(-\frac{1}{2}u_i+\frac{1}{2}u_j+u_k-u_l) & \frac{1}{4}(-v_i-v_j+v_k+v_l) & \frac{1}{2}(-u_i+u_k)+\frac{1}{3}\frac{B}{A}(-\frac{1}{2}v_i+\frac{1}{2}v_j+v_k-v_l) & 0 & \frac{1}{3}\frac{A}{B}(-\frac{1}{2}v_i-v_j+v_k+\frac{1}{2}v_l) & \frac{1}{4}(-v_i+v_j+v_k-v_l)+\frac{1}{3}\frac{A}{B}(-\frac{1}{2}u_i-u_j+u_k+\frac{1}{2}u_l) \\
0 & \frac{1}{4}(-u_i+u_j+u_k-u_l) & \frac{1}{3}\frac{B}{A}(-\frac{1}{2}u_i+\frac{1}{2}u_j+u_k-u_l) & \frac{1}{3}\frac{A}{B}(-\frac{1}{2}v_i-v_j+v_k+\frac{1}{2}v_l) & \frac{1}{2}(-v_i+v_k)+\frac{1}{3}\frac{A}{B}(-\frac{1}{2}u_i-u_j+u_k+\frac{1}{2}u_l) & \frac{1}{4}(-u_i-u_j+u_k+u_l)+\frac{1}{3}\frac{B}{A}(-\frac{1}{2}v_i+\frac{1}{2}v_j+v_k-v_l) \\
\frac{1}{3}\frac{B}{A}(\frac{1}{2}u_i-\frac{1}{2}u_j-u_k+u_l) & \frac{1}{4}(v_i+v_j-v_k-v_l) & \frac{1}{2}(u_j-u_l)+\frac{1}{3}\frac{B}{A}(\frac{1}{2}v_i-\frac{1}{2}v_j-v_k+v_l) & 0 & \frac{1}{3}\frac{A}{B}(-v_i-\frac{1}{2}v_j+\frac{1}{2}v_k+v_l) & \frac{1}{4}(-v_i+v_j+v_k-v_l)+\frac{1}{3}\frac{A}{B}(-u_i-\frac{1}{2}u_j+\frac{1}{2}u_k+u_l) \\
0 & \frac{1}{4}(-u_i+u_j+u_k-u_l) & \frac{1}{3}\frac{B}{A}(\frac{1}{2}u_i-\frac{1}{2}u_j-u_k+u_l) & \frac{1}{3}\frac{A}{B}(-v_l-\frac{1}{2}v_j+\frac{1}{2}v_k+v_l) & \frac{1}{2}(v_j-v_l)+\frac{1}{3}\frac{A}{B}(-u_i-\frac{1}{2}u_j+\frac{1}{2}u_k+u_l) & \frac{1}{4}(u_i+u_j-u_k-u_l)+\frac{1}{3}\frac{B}{A}(\frac{1}{2}v_i-\frac{1}{2}v_j-v_k+v_l)
\end{bmatrix}
$$

FIGURE 8 : $[k^*]^e$, MODIFIED ELEMENTAL STIFFNESS MATRIX FOR A 4-NODED RECTANGULAR FINITE ELEMENT, CLOSED-FORM SOLUTION IN TERMS OF ELEMENT DIMENSIONS (A,B) AND NODAL DISPLACEMENTS (u,v).

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CHARACTERIZATION OF ELASTIC SOLIDS USING FINITE ELEMENT METHODS

U. S. NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA
DECEMBER 1972


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C    ************************************************************************00000010
CC   THIS PROGRAM PERFORMS A PLANE STRESS OR A PLANE STRAIN ANALYSIS      00000020
CC   BY THE FINITE ELEMENT METHOD. NUMERICALLY INTEGRATED ISOPARAMETRIC   00000030
CC   ELEMENTS ARE USED THROUGHOUT.                                        00000040
CC   PROFESSOR GILLES CANTIN,  MARCH 1972                                 00000050
C    ************************************************************************00000060
       IMPLICIT REAL*8 (A-H,O-Z)                                          00000070
       COMMON/SOL/BGK(312,64),ALOAD(312),DISP(312),ABGN                   00000080
       COMMON/INT/NEL,NJT,NMAT,NCLOAD,NPBC,NCON(100,14),NBC(156,2),NSTRES00000090
      1,NGP,LM(22),LJT(22)                                                00000100
       COMMON/FLPL/COARD(156,2),CLOAD(156,2),ELCON(10,2),TITLE(20),T      00000110
C                                                                          00000120
CC   INPUT CARDS NEEDED                                                   00000130
CC                                                                         00000140
CC   TITLE                          (20A4)          ONE CARD PER PROBLEM  00000150
CC   PROBLEM PARAMETERS     (7I5,10X,1G24.16)  ONE CARD PER PROBLEM       00000160
CC    COL.    VARIABLE                                                     00000170
CC     1-5       NEL  NUMBER OF ELEMENTS (MAX. IS 100)                    00000180
CC     6-10      NJT  NUMBER OF JOINTS (MAX. IS 156)                      00000190
CC    11-15      NMAT  NUMBER OF MATERIALS (MAX. IS 10)                   00000200
CC    16-20      NCLOAD  NUMBER OF CONCENTRATED LOADS                     00000210
CC    21-25      NPBC  NUMBER OF JOINTS WITH BOUNDARY CONDITIONS          00000220
CC    26-30      NSTRESS  ZERO FOR PLANE STRESS PROBLEMS, ONE FOR P.STRAIN 00000230
CC    31-35      NGP  NUMBER OF GAUSS POINTS DESIRED IN THE INTEGRATION   00000240
CC    46-70      T    SPECIMEN THICKNESS:  T=0.0 OR BLANK IS SAME AS      00000250
CC       UNIT THICKNESS (T=1.0)                    (1G24.16)               00000260
CC                                                                         00000270
CC   JOINT CARDS  (CARD ONE:1I10,2G31.16 CARD TWO:2G31.16)   NJT PER PROB.00000280
CC      TWO CARDS PER JOINT ARE REQUIRED:                                  00000290
CC       CARD ONE--JOINT IDENTIFICATION AND COORDINATES                    00000300
CC     1-10      JOINT NUMBER                                             00000310
CC    11-41      X  COORDINATE OF THE JOINT                               00000320
CC    42-72      Y  COORDINATE OF THE JOINT                               00000330
CC       CARD TWO--LOADING AT THE JOINT                                    00000340
CC    1-31       X COMPONENT OF THE LOAD AT THE JOINT                     00000350
CC    32-62      Y COMPONENT OF THE LOAD AT THE JOINT                     00000360
CC                                                                         00000370
```

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C  MATERIAL PROPERTIES     (1I10,2G25.16)  NMAT CARDS PER PROBLEM          00000380
CC  COL.   VARIABLE                                                       00000390
CC   1-10      MATERIAL NUMBER                                            00000400
C   11-35      YOUNG'S MODULUS                                            00000410
C   36-60      POISSON'S RATIO                                            00000420
      DIMENSION REACT(157,2),NCODE(156)                                   00000430
      EQUIVALENCE (BGK(1,1),REACT(1,1))                                   00000440
C                                                                         00000450
CC  ELEMENT CARDS              (15I4)          NEL  CARDS PER PROBLEM     00000460
CC  COL.   VARIABLE                                                       00000470
CC   1-4       ELEMENT IDENTIFICATION NUMBER                              00000480
CC   5-52      ELEMENT CONNECTIVITY (COUNTERCLOCKWISE)                    00000490
CC  53-56      KIND  1  FOR FOUR NODAL POINT ELEMENT                      00000500
CC                   2  FOR EIGHT NODAL POINT ELEMENT                    00000510
CC                   3  FOR TWELVE NODAL POINT ELEMENT                   00000520
CC  57-60      TYPE  THIS IS THE MATERIAL IDENTIFICATION NUMBER           00000530
CC                                                                        00000540
CC  BOUNDARY CONDITION CARDS  (2I5)          NPBC  CARDS PER PROBLEM      00000550
CC  COL.   VARIABLE                                                       00000560
CC   1-5   JOINT NUMBER                                                   00000570
CC   6-10 CODE  1 FOR A ROLLER ON AN X AXIS, DISP. IN Y DIRECT. IS ZERO   00000580
CC               2 FOR A ROLLER ON AN Y AXIS, DISP. IN X DIRECT. IS ZERO  00000590
CC               3 FOR A FIXED JOINT, BOTH DISP. COMPONENTS ARE ZERO      00000600
CC               4 FOR A ROLLER ON AN X AXIS, THE DISP. IN THE Y DIRECT.  00000610
CC            IS SPECIFIED ON THE SECOND JOINT CARD IN COLS. 32-62        00000620
CC               5 FOR A ROLLER ON A Y AXIS, THE DISP. IN THE X DIRECT.   00000630
CC            IS SPECIFIED ON THE SECOND JOINT CARD IN COLS. 1-31         00000640
CC               6 FIXED JOINT WITH DISP. SPECIFIED IN BOTH X AND Y       00000650
CC            DIRECTIONS AND PUNCHED IN COLS. 1-31 AND COLS. 32-62 OF     00000660
CC            THE SECOND JOINT CARD (JOINT LOADING)                       00000670
CC               7 FOR A ROLLER ON AN AXIS INCLINED ALPHA DEGREES         00000680
CC                 WITH RESPECT TO THE X AXIS. THE JOINT CARD MAY         00000690
CC                 CONTAIN A TANGENTIAL LOAD IN COL.  1-31 AND THE        00000700
CC                 ANGLE ALPHA DEGREES IN COL. 32-62, SECOND JOINT CARD   00000710
C  AS MANY PROBLEMS AS ONE DESIRES MAY BE STACKED, BUT THE LAST CARD OF    00000720
C  A RUN MUST CONTAIN THE WORD    STOP    IN  THE FIRST FOUR COLUMNS       00000730
      PI=3.14159265358979D0                                               00000740
   10 NBAND=0                                                             00000750
      DO 20 I=1,156                                                       00000760
   20 NCODE(I)=0                                                          00000770
      CALL INPUT                                                          00000780
      NEQ=2*NJT                                                           00000790
      NPEL=NCON(1,13)*4                                                   00000800
      DO 300 I=1,NEL                                                      00000810
      DO 100 J=1,NPEL                                                     00000820
  100 LM(J)=NCON(I,J)                                                     00000830
      NPELM=NPEL-1                                                        00000840
      DO 200 K=1,NPELM                                                    00000850
```

```
      JK=K+1                                                                    00000860
      DO 200 L=JK,NPEL                                                          00000870
  200 NBAND=MAXO(NBAND,IABS(LM(K)-LM(L)))                                       00000880
  300 CONTINUE                                                                  00000890
      NBAND=(NBAND+1)*2                                                         00000900
      DO 350 I=1,NEQ                                                            00000910
      ALOAD(I)=0.0D0                                                            00000920
      DO 350 J=1,NBAND                                                          00000930
  350 BGK(I,J)=0.0D0                                                            00000940
      CALL MERGE(NPEL,NEQ)                                                      00000950
      CALL BCOND(NBAND)                                                         00000960
      CALL CHOL(NEQ,NBAND)                                                      00000970
      DO 500 I=1,NPBC                                                           00000980
      IJT=NBC(I,1)                                                              00000990
  500 NCODE(IJT)=NBC(I,2)                                                       00001000
      DO 600 I=1,NJT                                                            00001010
      II=I*2-1                                                                  00001020
      KODE=NCODE(I)                                                             00001030
      IF(KODE.EQ.7) GO TO 560                                                   00001040
      IF((KODE.EQ.5).OR.(KODE.EQ.6)) GO TO 550                                  00001050
      ALOAD(II)=CLOAD(I,1)+ALOAD(II)                                            00001060
  550 II=II+1                                                                   00001070
      IF((KODE.EQ.4).OR.(KODE.EQ.6)) GO TO 600                                  00001080
      ALOAD(II)=CLOAD(I,2)+ALOAD(II)                                            00001090
      GO TO 600                                                                 00001100
  560 ALF=CLOAD(I,2)*PI/180.0D0                                                 00001110
      COSA=DCOS(ALF)                                                            00001120
      SINA=DSIN(ALF)                                                            00001130
      IIP=II+1                                                                  00001140
      A1=ALOAD(II)*COSA+ALOAD(IIP)*SINA+CLOAD(I,1)                              00001150
      A2=ALOAD(IIP)*COSA-ALOAD(II)*SINA                                         00001160
      ALOAD(II)=A1                                                              00001170
      ALOAD(IIP)=A2                                                             00001180
  600 CONTINUE                                                                  00001190
      CALL SOLV(NEQ,NBAND)                                                      00001200
  610 DO 630 I=1,NPBC                                                           00001210
      DO 630 J=1,2                                                              00001220
  630 REACT(I,J)=0.0D0                                                          00001230
      CALL DISPL                                                                00001240
      WRITE(6,1000)                                                             00001250
 1000 FORMAT(//,' DISPLACEMENTS',//,3X,' JOINT',10X,'  U',22X,'  V',/)          00001260
      DO 700 I=1,NJT                                                            00001270
      II=I*2-1                                                                  00001280
      III=II+1                                                                  00001290
  700 WRITE(6,2000) I,DISP(II),DISP(III)                                        00001300
 2000 FORMAT(2X,I3,2G25.16)                                                     00001310
      WRITE(6,3000)                                                             00001320
 3000 FORMAT(//,'     REACTIONS',//,3X,' JOINT',10X,'  RX',22X,'  RY',/)        00001330
```

```
      WRITE(6,2500) REACT(157,1),REACT(157,2)                          00001380
 2500 FORMAT(5X,2G25.16)                                               00001390
      CALL PUNCH(NJT,DISP)                                             00001400
      CALL STRESS(NPEL)                                                00001410
  890 GO TO 10                                                         00001420
      END                                                              00001430


      SUBROUTINE INPUT                                                 00001440
      IMPLICIT REAL*8 (A-H,O-Z)                                        00001450
      COMMON/INT/NEL,NJT,NMAT,NCLOAD,NPBC,NCON(100,14),NBC(156,2),NSTRES00001460
     1,NGP,LM(22),LJT(22)                                              00001470
      COMMON/FLPL/COARD(156,2),CLOAD(156,2),ELCON(10,2),TITLE(20),T    00001480
      DATA CHK /'STOP'/                                                00001490
      READ (5,1000) TITLE                                              00001500
 1000 FORMAT (20A4)                                                    00001510
      IF (TITLE(1).EQ.CHK) STOP                                        00001520
      DO 1050 I=1,156                                                  00001530
      DO 1050 J=1,2                                                    00001540
      NBC(I,J)=0                                                       00001550
      COARD(I,J)=0.0D0                                                 00001560
 1050 CLOAD(I,J)=0.0D0                                                 00001570
      DO 1060 I=1,10                                                   00001580
      DO 1060 J=1,2                                                    00001590
 1060 ELCON(I,J)=0.0D0                                                 00001600
      DO 1070 I=1,156                                                  00001610
      DO 1070 J=1,14                                                   00001620
 1070 NCON(I,J)=0                                                      00001630
      WRITE(6,1080) TITLE                                              00001640
 1080 FORMAT('1',20A4)                                                 00001650
C                                                                      00001660
C     READ STRUCTURE PARAMETERS                                        00001670
C                                                                      00001680
      READ (5,1100) NEL,NJT,NMAT,NCLOAD,NPBC,NSTRES,NGP,T              00001690
 1100 FORMAT (7I5,10X,1G24.16)                                         00001700
      IF (T.GT.0.0) GO TO 1150                                         00001710
      T=1.0D0                                                          00001720
 1150 WRITE (6,1200) NEL,NJT,NMAT,NCLOAD,NPBC,NSTRES,NGP,T             00001730
 1200 FORMAT (//,' NUMBER OF ELEMENTS=',I5,//,' TOTAL NUMBER OF JOINTS='00001740
     1,I5,//,' NUMBER OF MATERIALS=',I5,//,' NUMBER OF CONCENTRATED LOAD00001750
     2S=',I5,//,' NUMBER OF JOINTS WITH BOUNDARY CONDITIONS=',I5,//,' TH00001760
     3E VALUE OF NSTRES=',I5,//,' NUMBER OF GAUSS POINTS USED ',I5,     00001770
     4//,' ',' SPECIMEN IS OF THICKNESS',1G24.16)                      00001780
C                                                                      00001790
```

NPBC1=(NPBC/2)+1

```
      NPBCM=NPBC-1                                                      00002280
      DO 4000 K=1,NPBCT                                                 00002290
      DO 4000 I=1,NPBCM                                                 00002300
      IF(NBC(I,2).EQ.7) GO TO 3600                                      00002310
      GO TO 4000                                                        00002320
 3600 N1=NBC(I,1)                                                       00002330
      DO 3800 J=I,NPBCM                                                 00002340
      NBC(J,1)=NBC(J+1,1)                                               00002350
 3800 NBC(J,2)=NBC(J+1,2)                                               00002360
      NBC(NPBC,1)=N1                                                    00002370
      NBC(NPBC,2)=7                                                     00002380
 4000 CONTINUE                                                          00002390
      RETURN                                                            00002400
      END                                                               00002410

      SUBROUTINE MERGE(NPEL,NEQ)                                        00002420
      IMPLICIT REAL*8 (A-H,O-Z)                                         00002430
      DIMENSION          STK(24,24),AK(24,24),B(3,24)                   00002440
      COMMON/SOL/BGK(312,64),ALOAD(312),DISP(312),ABGN                  00002450
      COMMON/INT/NEL,NJT,NMAT,NCLOAD,NPBC,NCON(100,14),NBC(156,2),NSTRES00002460
     1,NGP,LM(22),LJT(22)                                               00002470
      COMMON/FLPL/COARD(156,2),CLOAD(156,2),ELCON(10,2),TITLE(20),T     00002480
      COMMON COORD(12,2),ELAST(3,3),SS(36,24),SN(36,24)                 00002490
      REWIND 10                                                         00002500
      DO 100 I=1,36                                                     00002510
      DO 100 J=1,24                                                     00002520
      SS(I,J)=0.0D0                                                     00002530
  100 SN(I,J)=0.0D0                                                     00002540
      DO 200 IK=1,NEL                                                   00002550
      DO 110 NN=1,NPEL                                                  00002560
      LJT(NN)=NCON(IK,NN)                                               00002570
  110 LM(NN)=(NCON(IK,NN)-1)*2                                          00002580
      NS=3*NPEL                                                         00002590
      DO 120 I1=1,NPEL                                                  00002600
      I2=LJT(I1)                                                        00002610
      DO 120 J1=1,2                                                     00002620
  120 COORD(I1,J1)=COARD(I2,J1)                                         00002630
      MATN=NCON(IK,14)                                                  00002640
      N=NCON(IK,13)*8                                                   00002650
      YM=ELCON(MATN,1)                                                  00002660
      PRTIO=ELCON(MATN,2)                                               00002670
      CALL ELASM(ELAST,YM,PRTIO,NSTRES,T)                               00002680
      CALL QUAD5(STK,AK,B,NS,N,NGP)                                     00002690
      WRITE(10) SS,SN                                                   00002700
      DO 200 I=1,NPEL                                                   00002710
      DO 200 J=1,NPEL                                                   00002720
      DO 200 K=1,2                                                      00002730
```

```
      II=LM(I)+K                                                        00002740
      KK=2*(I-1)+K                                                      00002750
      DO 200 L=1,2                                                      00002760
      JJ=LM(J)+L-II+1                                                   00002770
      IF(JJ.LE.0) GO TO 200                                             00002780
      LL=2*(J-1)+L                                                      00002790
      BGK(II,JJ)=BGK(II,JJ)+STK(KK,LL)                                  00002800
  200 CONTINUE                                                          00002810
      SUM=0.0D0                                                         00002820
      DO 300 I=1,NEQ                                                    00002830
  300 SUM=SUM+BGK(I,1)                                                  00002840
      ABGN=SUM*1.0D20                                                   00002850
      RETURN                                                            00002860
      END                                                               00002870

      SUBROUTINE ELASM(D,E,PR,NN,TH)                                    00002880
      IMPLICIT REAL*8 (A-H,O-Z)                                         00002890
C     THIS SUBROUTINE ASSEMBLYS THE ELASTIC MATRIX FOR PLANE            00002900
C     STRAIN (NN=1) OR PLANE STRESS (NN=0)                              00002910
      DIMENSION D(3,3)                                                  00002920
      DO 20 I=1,3                                                       00002930
      DO 20 J=1,3                                                       00002940
   20 D(I,J)=0.0D0                                                      00002950
      IF(NN.EQ.0) GO TO 10                                              00002960
C     FORM ELASTIC MATRIX FOR PLANE STRAIN                              00002970
      ER=TH*E/((1.0D0+PR)*(1.0D0-2.0D0*PR))                             00002980
      D(1,1)=ER*(1.0D0-PR)                                              00002990
      D(1,2)=ER*PR                                                      00003000
      D(2,1)=D(1,2)                                                     00003010
      D(2,2)=D(1,1)                                                     00003020
      D(3,3)=ER*(0.5D0-PR)                                              00003030
      RETURN                                                            00003040
C     FORM ELASTIC MATRIX FOR PLANE STRESS                              00003050
   10 ER=TH*E/(1.0D0-PR*PR)                                             00003060
      D(1,1)=ER                                                         00003070
      D(1,2)=ER*PR                                                      00003080
      D(2,1)=D(1,2)                                                     00003090
      D(2,2)=D(1,1)                                                     00003100
      D(3,3)=ER*(1.0D0-PR)/2.0D0                                        00003110
      RETURN                                                            00003120
      END                                                               00003130
```

```
      SUBROUTINE QUAD5(STK,AK,B,NS,N,NGP)                                   00003140
      IMPLICIT REAL*8 (A-H,O-Z)                                             00003150
C                                                                           00003160
C     QUAD5 IS A NUMERICAL INTEGRATION SUBROUTINE THAT FORMS THE STIFFNESS  00003170
C     MATRIX FOR ANY OF THE THREE QUADRILATERAL ELEMENTS IN THIS FAMILY.    00003180
C     GAUSSIAN QUADRATURE WITH A MAXIMUM OF FIVE ORDINATES IS USED.         00003190
C                                                                           00003200
C     THE SUBROUTINE IS CALLED WITH THE FOLLOWING STATEMENT:                00003210
C     CALL QUAD5(STK,AK,SS,SN,NS,N,NGP)                                     00003220
C     THE CALLING PROGRAM MUST CONTAIN THE FOLLOWING COMMON STATEMENT:      00003230
C     COMMON COORD(12,2),ELAST(3,3)                                         00003240
C          STK   IS AN ARRAY DIMENSIONED NXN THAT WILL CONTAIN THE STIFFNES 00003250
C                MATRIX UPON RETURN TO THE CALLING PROGRAM                  00003260
C          SS AND SN ARE THE STRESS AND STRAIN TRANSFORMATION MATRICES,     00003270
C                VERSES NODAL POINT DISPLACEMENTS; BOTH ARE DIMENSIONED     00003280
C                NSXN--NS IS 3*NPT, N IS 2*NPT. SCRATCH ACCESS ON UNIT 10.  00003290
C          AK    IS AN ARRAY DIMENSIONED NXN REQUIRED IN FORMK              00003300
C          B     IS AN ARRAY DIMENSIONED 2XN REQUIRED IN FORMK              00003310
C                                                                           00003320
C          NGP IS THE NUMBER OF GAUSS POINTS UED IN THE INTEGRATION         00003330
C                                                                           00003340
C          COORD(12,2)  IS AN ARRAY CONTAINING THE COORDINATES OF ALL       00003350
C                       THE NODAL POINTS OF THE ELEMENT TO BE CONSTRUCTED   00003360
C          ELAST(3,3)   IS AN ARRAY CONTAINING ALL THE ELASTIC CONSTANTS    00003370
C                       OF THE ELEMENT, THE ORDER OF THE STRESS COMPONENTS  00003380
C                       IS: SIGMAX,SIGMAY,TAUXY                             00003390
C                                                                           00003400
      DIMENSION AK(24,24),STK(24,24),B(3,24)                                00003410
      DIMENSION XI(5),AI(5),AIA(5,5)                                        00003420
      DIMENSION X2(2),X3(3),X4(4),X5(5),A2(2),A3(3),A4(4),A5(5)             00003430
      DIMENSION XYL(4,2),XYQ(8,2),XYC(12,2)                                 00003440
C                                                                           00003450
C           ETA                    ETA                    ETA              00003460
C            +                      +                      +               00003470
C     U2----+-----U1         U3----U2----U1         U4--U3+-U2--U1         00003480
C     |      +     |         |      +     |         |      +     |         00003490
C     |      +     |         |      +     |         |      +     |         00003500
C     |      +     |         |      +     |         |      +     |         00003510
C     |      +     |         |      +     |         U5     +     U12       00003520
C     |      +     |         |      +     |         |      +     |         00003530
C     |      ++++++++XI      U4     ++++++U8++XI    |      ++++++++XI      00003540
C     |            |         |            |         |            |         00003550
C     |            |         |            |         U6           U11       00003560
C     |            |         |            |         |            |         00003570
C     |            |         |            |         |            |         00003580
C     |            |         |            |         |            |         00003590
C     U3----------U4         U5----U6----U7         U7--U8--U9--U10        00003600
C                                                                           00003610
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C   CODED BY PROFESSOR GILLES CANTIN AT THE NAVAL POSTGRADUATE SCHOOL   00003620
C                                                                       00003630
      COMMON COORD(12,2),ELAST(3,3),SS(36,24),SN(36,24)                 00003640
      DATA X2/0.57735026918962 6D0,-0.57735026918962 6D0/               00003650
      DATA A2/1.0D0,1.0D0/                                              00003660
      DATA X3/0.77459666924148 3D0,0.0D0,-0.77459666924148 3D0/         00003670
      DATA A3/0.55555555555555 6D0,0.88888888888888 9D0,0.55555555555555 00003680
     156D0/                                                             00003690
      DATA X4/0.86113631159405 3D0,0.33998104358485 6D0,-0.33998104358485600003700
     1D0,-0.86113631159405 3/                                           00003710
      DATA A4/0.34785484513745 4D0,0.65214515486254 6D0,0.65214515486254 6D00003720
     10,0.34785484513745 4D0/                                           00003730
      DATA X5/0.90617984593866 4D0,0.53846931010568 3D0,0.0D0,-0.5538469100003740
     10105683D0,-0.90617984593866 4D0/                                  00003750
      DATA A5/0.23692688505618 9D0,0.47862867049936 6D0,0.56888888888888 900003760
     1D0,0.47862867049936 6D0,0.23692688505618 9D0/                     00003770
      DATA XYL/1.0D0,-1.0D0,-1.0D0,1.0D0,1.0D0,1.0D0,-1.0D0,-1.0D0/     00003780
      DATA XYQ/1.0D0,0.0D0,-1.0D0,-1.0D0,-1.0D0,0.0D0,1.0D0,1.0D0,1.0D0,00003790
     11.0D0,1.0D0,0.0D0,-1.0D0,-1.0D0,-1.0D0,0.0D0/                     00003800
      DATA XYC/1.0D0,0.33333333333333 3D0,-0.33333333333333 3D0,-1.0D0, 00003810
     1 -1.0D0,-1.0D0,-1.0D0,-0.33333333333333 3D0,0.33333333333333 3D0, 00003820
     11.0D0,1.0D0,1.0D0,1.0D0,1.0D0,1.0D0,1.0D0,0.33333333333333 3D0,   00003830
     1-0.33333333333333 3D0,-1.0D0,-1.0D0,-1.0D0,-1.0D0,-0.33333333333330003840
     1333D0,0.33333333333333 3D0/                                       00003850
      IF(NGP.EQ.0) NGP=5                                                00003860
      DO 100 I=1,N                                                      00003870
      DO 100 J=1,N                                                      00003880
  100 STK(I,J)=0.0D0                                                    00003890
      JUMP=NGP-1                                                        00003900
      DO 140 I=1,NGP                                                    00003910
      GO TO (132,133,134,135),JUMP                                      00003920
  132 XI(I)=X2(I)                                                       00003930
      AI(I)=A2(I)                                                       00003940
      GO TO 140                                                         00003950
  133 XI(I)=X3(I)                                                       00003960
      AI(I)=A3(I)                                                       00003970
      GO TO 140                                                         00003980
  134 XI(I)=X4(I)                                                       00003990
      AI(I)=A4(I)                                                       00004000
      GO TO 140                                                         00004010
  135 XI(I)=X5(I)                                                       00004020
      AI(I)=A5(I)                                                       00004030
  140 CONTINUE                                                          00004040
      DO 150 I=1,NGP                                                    00004050
      DO 150 J=1,NGP                                                    00004060
  150 AIA(I,J)=AI(I)*AI(J)                                              00004070
      DO 200 I=1,NGP                                                    00004080
      X=XI(I)                                                           00004090
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      DO 200 J=1,NGP                                                    00004100
      Y=XI(J)                                                           00004110
      CALL FORMK(AK,X,Y,B,N)                                            00004120
      DO 200 K=1,N                                                      00004130
      DO 200 L=1,N                                                      00004140
  200 STK(K,L)=STK(K,L)+AIA(I,J)*AK(K,L)                                00004150
      DO 300 I=1,N                                                      00004160
      DO 300 J=I,N                                                      00004170
      STK(I,J)=(STK(I,J)+STK(J,I))*0.5D0                                00004180
  300 STK(J,I)=STK(I,J)                                                 00004190
      NPT=N/2                                                           00004200
      IGO=NPT/4                                                         00004210
      DO 1000 I=1,NPT                                                   00004220
      GO TO (400,500,600),IGO                                           00004230
  400 X=XYL(I,1)                                                        00004240
      Y=XYL(I,2)                                                        00004250
      GO TO 700                                                         00004260
  500 X=XYQ(I,1)                                                        00004270
      Y=XYQ(I,2)                                                        00004280
      GO TO 700                                                         00004290
  600 X=XYC(I,1)                                                        00004300
      Y=XYC(I,2)                                                        00004310
  700 CALL FORMK(AK,X,Y,B,N)                                            00004320
      DO 800 L=1,3                                                      00004330
      DO 800 M=1,N                                                      00004340
      AK(L,M)=0.0D0                                                     00004350
      DO 800 NN=1,3                                                     00004360
  800 AK(L,M)=AK(L,M)+ELAST(L,NN)*B(NN,M)                               00004370
      II=3*(I-1)                                                        00004380
      DO 900 J=1,3                                                      00004390
      IJ=II+J                                                           00004400
      DO 900 K=1,N                                                      00004410
      SS(IJ,K)=AK(J,K)                                                  00004420
  900 SN(IJ,K)=B(J,K)                                                   00004430
 1000 CONTINUE                                                          00004440
      RETURN                                                            00004450
      END                                                               00004460

      SUBROUTINE FORMK(AK,X,Y,B,N)                                      00004470
      IMPLICIT REAL*8 (A-H,O-Z)                                         00004480
C                                                                       00004490
C                                                                       00004500
C THIS SUBROUTINE FORMS THE STIFFNESS AND THE B MATRIX AS FUNCTIONS OF  00004510
C XI AND ETA FOR THE THREE DIFFERENT QUADRILATERAL ELEM. IN THIS FAMILY 00004520
C                                                                       00004530
C                                                                       00004540
      DIMENSION AJ(2,2),AJIN(2,2),DNX(2,12),W1(2,12),B(3,24)            00004550
```

```
     1,B1(24,3),AK(24,24)                                                    00004560
      COMMON COORD(12,2),ELAST(3,3),SS(36,24),SN(36,24)                      00004570
C INITIALIZE                                                                 00004580
      DO 10 I=1,3                                                            00004590
      DO 10 J=1,N                                                            00004600
   10 B(I,J)=0.0D0                                                           00004610
      ZERO=0.0D0                                                             00004620
      ONE=1.0D0                                                              00004630
      TWO=2.0D0                                                              00004640
      FOUR=4.0D0                                                             00004650
      NPT=N/2                                                                00004660
C                                                                            00004670
C FORM (2XNPT) MATRIX OF DERIV. OF THE INTERP. FUNCT. WRT XI AND ETA         00004680
C                                                                            00004690
      IGO=N/8                                                                00004700
      GO TO (20,30,40),IGO                                                   00004710
C                                                                            00004720
C    LINEAR FUNCTIONS                                                        00004730
C                                                                            00004740
   20 W1(1,3)=-(ONE-Y)/FOUR                                                  00004750
      W1(1,4)=-W1(1,3)                                                       00004760
      W1(1,1)= (ONE+Y)/FOUR                                                  00004770
      W1(1,2)=-W1(1,1)                                                       00004780
      W1(2,3)=-(ONE-X)/FOUR                                                  00004790
      W1(2,4)=-(ONE+X)/FOUR                                                  00004800
      W1(2,1)=-W1(2,4)                                                       00004810
      W1(2,2)=-W1(2,3)                                                       00004820
      GO TO 50                                                               00004830
   30 CONTINUE                                                               00004840
C                                                                            00004850
C    QUADRATIC FUNCTIONS                                                     00004860
C                                                                            00004870
      TXPY=TWO*X+Y                                                           00004880
      TXMY=TWO*X-Y                                                           00004890
      TYPX=TWO*Y+X                                                           00004900
      TYMX=TWO*Y-X                                                           00004910
      OMY=ONE-Y                                                              00004920
      OPY=ONE+Y                                                              00004930
      OPX=ONE+X                                                              00004940
      OMX=ONE-X                                                              00004950
      W1(1,5)=OMY*TXPY/FOUR                                                  00004960
      W1(1,6)=-OMY*X                                                         00004970
      W1(1,7)=OMY*TXMY/FOUR                                                  00004980
      W1(1,8)=OPY*OMY/TWO                                                    00004990
      W1(1,1)=OPY*TXPY/FOUR                                                  00005000
      W1(1,2)=-OPY*X                                                         00005010
      W1(1,3)= OPY*TXMY/FOUR                                                 00005020
      W1(1,4)=-OPY*OMY/TWO                                                   00005030
```

```
      W1(2,5)=OMX*TYPX/FOUR                                              00005040
      W1(2,6)=-OPX*OMX/TWO                                               00005050
      W1(2,7)=OPX*TYMX/FOUR                                              00005060
      W1(2,8)=-Y*OPX                                                     00005070
      W1(2,1)=OPX*TYPX/FOUR                                              00005080
      W1(2,2)=OPX*OMX/TWO                                                00005090
      W1(2,3)=OMX*TYMX/FOUR                                              00005100
      W1(2,4)=-Y*OMX                                                     00005110
      GO TO 50                                                           00005120
   40 CONTINUE                                                           00005130
C                                                                        00005140
CC    CUBIC FUNCTIONS                                                    00005150
C                                                                        00005160
      OPX=ONE+X                                                          00005170
      OPY=ONE+Y                                                          00005180
      OMX=ONE-X                                                          00005190
      OMY=ONE-Y                                                          00005200
      TPTMNX=3.0D0+2.0D0*X-9.0D0*X*X                                     00005210
      TMTMNX=3.0D0-2.0D0*X-9.0D0*X*X                                     00005220
      TPTMNY=3.0D0+2.0D0*Y-9.0D0*Y*Y                                     00005230
      TMTMNY=3.0D0-2.0D0*Y-9.0D0*Y*Y                                     00005240
      TYPO=3.0D0*Y+ONE                                                   00005250
      TYMO=3.0D0*Y-ONE                                                   00005260
      TXPO=3.0D0*X+ONE                                                   00005270
      TXMO=3.0D0*X-ONE                                                   00005280
      TMTPEX=10.0D0-27.0D0*X*X+18.0D0*X-9.0D0*Y*Y                        00005290
      TMTMEX=10.0D0-27.0D0*X*X-18.0D0*X-9.0D0*Y*Y                        00005300
      TMNPEY= 10.0D0-9.0D0*X*X+18.0D0*Y-27.0D0*Y*Y                       00005310
      TMNMEY= 10.0D0-9.0D0*X*X-18.0D0*Y-27.0D0*Y*Y                       00005320
      TTT=32.0D0                                                         00005330
      TTN=32.0D0/9.0D0                                                   00005340
      W1(1,7)= OMY*TMTPEX/TTT                                            00005350
      W1(1,8)=-TPTMNX*OMY/TTN                                            00005360
      W1(1,9)=TMTMNX*OMY/TTN                                             00005370
      W1(1,10)=-OMY*TMTMEX/TTT                                           00005380
      W1(1,11)=-OPY*OMY*TYMO/TTN                                         00005390
      W1(1,12)=OPY*OMY*TYPO/TTN                                          00005400
      W1(1,1)=-OPY*TMTMEX/TTT                                            00005410
      W1(1,2)=TMTMNX*OPY/TTN                                             00005420
      W1(1,3)=-TPTMNX*OPY/TTN                                            00005430
      W1(1,4) =OPY*TMTPEX/TTT                                            00005440
      W1(1,5) =-OPY*OMY*TYPO/TTN                                         00005450
      W1(1,6) = OPY*OMY*TYMO/TTN                                         00005460
      W1(2,7) = OMX*TMNPEY/TTT                                           00005470
      W1(2,8)= OPX*OMX*TXMO/TTN                                          00005480
      W1(2,9)=-OPX*OMX*TXPO/TTN                                          00005490
      W1(2,10) = OPX*TMNPEY/TTT                                          00005500
      W1(2,11)=-TPTMNY*OPX/TTN                                           00005510
```

```
      W1(2,12)= TMTMNY*OPX/TTN                                          00005520
      W1(2,1) =-OPX*TMNMEY/TTT                                          00005530
      W1(2,2)=OPX*OMX*TXPO/TTN                                          00005540
      W1(2,3)=-OPX*OMX*TXMO/TTN                                         00005550
      W1(2,4) =-OMX*TMNMEY/TTT                                          00005560
      W1(2,5) =OMX*TMTMNY/TTN                                           00005570
      W1(2,6) =-OMX*TPTMNY/TTN                                          00005580
   50 CONTINUE                                                          00005590
C                                                                       00005600
C  FORM JACOBIAN OF THE TRANSFORMATION IN AJ(I,J)                       00005610
C                                                                       00005620
      DO 100 I=1,2                                                      00005630
      DO 100 J=1,2                                                      00005640
      AJ(I,J)=ZERO                                                      00005650
      DO 100 K=1,NPT                                                    00005660
  100 AJ(I,J)=AJ(I,J)+W1(I,K)*COORD(K,J)                                00005670
C                                                                       00005680
C  CALCULATE DETERMINANT OF THE JACOBIAN                                00005690
C                                                                       00005700
      DTJ=AJ(1,1)*AJ(2,2)-AJ(1,2)*AJ(2,1)                               00005710
C                                                                       00005720
C  INVERT JACOBIAN IN AJIN                                              00005730
C                                                                       00005740
      AJIN(1,1)=AJ(2,2)/DTJ                                             00005750
      AJIN(1,2)=-AJ(1,2)/DTJ                                            00005760
      AJIN(2,1)=-AJ(2,1)/DTJ                                            00005770
      AJIN(2,2)=AJ(1,1)/DTJ                                             00005780
C                                                                       00005790
C  FORM (2XNPT) MATRIX OF DERIVATIVES OF INTERP. FUNCT WRT X AND Y      00005800
C                                                                       00005810
      DO 200 I=1,2                                                      00005820
      DO 200 J=1,NPT                                                    00005830
      DNX(I,J)=ZERO                                                     00005840
      DO 200 K=1,2                                                      00005850
  200 DNX(I,J)=DNX(I,J)+AJIN(I,K)*W1(K,J)                               00005860
C                                                                       00005870
C  FORM B(I,J) MATRIX                                                   00005880
C                                                                       00005890
      DO 300 I=1,NPT                                                    00005900
      IOD=2*I-1                                                         00005910
      IEV=2*I                                                           00005920
      B(1,IOD)=DNX(1,I)                                                 00005930
      B(2,IEV)=DNX(2,I)                                                 00005940
      B(3,IEV)=DNX(1,I)                                                 00005950
  300 B(3,IOD)=DNX(2,I)                                                 00005960
C                                                                       00005970
C  FORM STIFFNESS BY THE CONGRUENT TRANSFORMATION  BT*D*B               00005980
C                                                                       00005990
```

```
      DO 500 I=1,N                                                          00006000
      DO 500 J=1,3                                                          00006010
      B1(I,J)=ZERO                                                          00006020
      DO 500 K=1,3                                                          00006030
  500 B1(I,J)=B1(I,J)+B (K,I)*ELAST(K,J)                                    00006040
      DO 600 I=1,N                                                          00006050
      DO 600 J=1,N                                                          00006060
      AK(I,J)=ZERO                                                          00006070
      DO 600 K=1,3                                                          00006080
  600 AK(I,J)=AK(I,J)+B1(I,K)*B(K,J)                                        00006090
      DO 700 I=1,N                                                          00006100
      DO 700 J=I,N                                                          00006110
      AK(I,J)=((AK(I,J)+AK(J,I))/2.0D0)*DTJ                                 00006120
  700 AK(J,I)=AK(I,J)                                                       00006130
      RETURN                                                                00006140
      END                                                                   00006150

      SUBROUTINE CHOL(NEQ,NBAND)                                            00006160
      IMPLICIT REAL*8 (A-H,O-Z)                                             00006170
C*********************************************************************    00006180
C     THIS SUBROUTINE DOES THE CHOLESKI DECOMPOSITION OF A MATRIX  (A)      00006190
C     WHEN  (A) IS SYMMETIRC AND BANDED,  (A) = (L)*(LT)                    00006200
C     THE MATRIX (A) MUST BE FURNISHED IN RECTANGULAR FORM A(NEQ,NBAND)     00006210
C     WITH ALL ITS DIAGONAL ELEMENTS IN THE FIRST COLUMN                    00006220
C     NEQ  IS THE TOTAL NUMBER OF EQUATIONS                                 00006230
C     NBAND IS THE HALF-BANDWIDTH OF THE SYSTEM                             00006240
C      CODED BY PROF. CANTIN, DECEMBER 1971                                 00006250
C*********************************************************************    00006260
C                                                                           00006270
      COMMON/SOL/ A(312,64),B(312),X(312),ABGN                              00006280
      ZRO=0.0D0                                                             00006290
      ANEQ=NEQ                                                              00006300
      APZRO=ABGN*1.0D-25/ANEQ                                               00006310
      DO 300 I=1,NEQ                                                        00006320
      DIAG=A(I,1)                                                           00006330
      IF(DIAG.LE.ZRO) GO TO 400                                             00006340
      IF(DIAG.LE.APZRO) WRITE(6,1000) I                                     00006350
 1000 FORMAT(//5X,' SYSTEM IS NEARLY SINGULAR AT EQUATION ',I5//)           00006360
      DIAG=DSQRT(DIAG)                                                      00006370
      DO 200 J=1,NBAND                                                      00006380
  200 A(I,J)=A(I,J)/DIAG                                                    00006390
      DO 260 J=2,NBAND                                                      00006400
      L=I+J-1                                                               00006410
      IF(L.GT.NEQ) GO TO 260                                                00006420
      AA=A(I,J)                                                             00006430
      IF(AA.EQ.ZRO) GO TO 260                                               00006440
      DO 250 K=J,NBAND                                                      00006450
```

```
      M=1+K-J                                                           00006460
  250 A(L,M)=A(L,M)-AA*A(I,K)                                           00006470
  260 CONTINUE                                                          00006480
  300 CONTINUE                                                          00006490
      DNM=0.0D0                                                         00006500
      DO 350 I=1,NEQ                                                    00006510
  350 DNM=DNM+1.0D0/A(I,1)                                              00006520
      ANUM=ABGN*1.0D-20                                                 00006530
      CNM=ANUM/DNM                                                      00006540
      WRITE(6,2000) ANUM,DNM,CNM                                        00006550
 2000 FORMAT(//,5X,' CONDITION NUMBER = (',1G24.16,')/('1G24.16,')=',   00006560
     11G22.16,//)                                                       00006570
      RETURN                                                            00006580
  400 WRITE(6,1100) I,DIAG                                              00006590
 1100 FORMAT(//5X,' SYSTEM IS SINGULAR AT EQUATION ',I5,/,5X,' OR INDEFI00006600
     1NITE, DIAG = ',1PG24.16,//)                                       00006610
      STOP                                                              00006620
      END                                                               00006630

      SUBROUTINE SOLV(NEQ,NBAND)                                        00006640
      IMPLICIT REAL*8 (A-H,O-Z)                                         00006650
C*******************************************************************00006660
C     THIS SUBROUTINE PERFORMS A FORWARD SUBSTITUTION FOLLOWED BY A     00006670
C     BACKWARD SUBSTITUTION IN THE SYSTEM  (A)*(X) = (B) WHERE THE MATRIX00006680
C     (A) HAS ALREADY BEEN DECOMPOSED BY THE SUBROUTINE CHOL            00006690
C     (A) IS THE MATRIX OF COEEFICIENTS AFTER A CALL TO S.R.CHOL        00006700
C     X IS A VECTOR THAT CONTAINS THE ANSWERS UPON EXIT FROM THE S.R.   00006710
C     NEQ AND NBAND ARE THE SAME AS IN CHOL                             00006720
C                                                                       00006730
C     AFTER ONE CALL TO S.R.CHOL MANY CALLS TO SOLV CAN BE MADE WITH    00006740
C     DIFFERENT VALUES FOR THE VECTOR (B) ,(B) IS DESTROYED EVERY TIME  00006750
C      CODED BY PROF. CANTIN, DECEMBER 1971                             00006760
C*******************************************************************00006770
C                                                                       00006780
      COMMON/SOL/ A(312,64),B(312),X(312),ABGN                          00006790
      ZRO=0.0D0                                                         00006800
      DO 200 I=1,NEQ                                                    00006810
      IF(B(I).EQ.ZRO) GO TO 200                                         00006820
      B(I)=B(I)/A(I,1)                                                  00006830
      DO 100 J=2,NBAND                                                  00006840
      L=I+J-1                                                           00006850
      IF(L.GT.NEQ) GO TO 100                                            00006860
      AA=A(I,J)                                                         00006870
      B(L)=B(L)-AA*B(I)                                                 00006880
  100 CONTINUE                                                          00006890
  200 CONTINUE                                                          00006900
      X(NEQ)=B(NEQ)/A(NEQ,1)                                            00006910
```

```
      DO 400 L=2,NEQ                                                    00006920
      K=NEQ-L+1                                                         00006930
      SUM=ZRO                                                           00006940
      DO 300 J=2,NBAND                                                  00006950
      M=J+K-1                                                           00006960
      IF(NEQ.LT.M) GO TO 300                                            00006970
      SUM=SUM+A(K,J)*X(M)                                               00006980
  300 CONTINUE                                                          00006990
  400 X(K)=(B(K)-SUM)/A(K,1)                                            00007000
      RETURN                                                            00007010
      END                                                               00007020

      SUBROUTINE STRESS(NPEL)                                           00007030
      IMPLICIT REAL*8 (A-H,O-Z)                                         00007040
      COMMON/SOL/BGK(312,64),ALOAD(312),DISP(312),ABGN                  00007050
      COMMON/INT/NEL,NJT,NMAT,NCLOAD,NPBC,NCON(100,14),NBC(156,2),NSTRES00007060
     1,NGP,LM(22),LJT(22)                                               00007070
      COMMON/FLPL/COARD(156,2),CLOAD(156,2),ELCON(10,2),TITLE(20),T     00007080
      COMMON COORD(12,2),ELAST(3,3),SS(36,24),SN(36,24)                 00007090
      DIMENSION SSJNT(100,7),SNJNT(100,7),SSEL(36),SNEL(36),DSPEL(24)   00007100
      EQUIVALENCE (BGK(1,1),SSJNT(1,1)),(BGK(1,5),SNJNT(1,1)),(BGK(1,9),00007110
     1SSEL(1)),(BGK(1,10),SNEL(1)),(BGK(1,11),DSPEL(1))                 00007120
      PI=3.14159265358979 3D0                                           00007130
      REWIND 10                                                         00007140
      DO 50 I=1,NJT                                                     00007150
      DO 50 J=1,7                                                       00007160
      SNJNT(I,J)=0.0D0                                                  00007170
   50 SSJNT(I,J)=0.0D0                                                  00007180
      DO 300 I=1,NEL                                                    00007190
      READ(10) SS,SN                                                    00007200
      DO 100 J=1,NPEL                                                   00007210
      JA=NCON(I,J)*2-1                                                  00007220
      JJ=J*2-1                                                          00007230
      DSPEL(JJ)=DISP(JA)                                                00007240
  100 DSPEL(JJ+1)=DISP(JA+1)                                            00007250
      NJLM=NPEL*2                                                       00007260
      NPELT=NPEL*3                                                      00007270
      DO 200 I1=1,NPELT                                                 00007280
      SSEL(I1)=0.0D0                                                    00007290
      SNEL(I1)=0.0D0                                                    00007300
      DO 200 J1=1,NJLM                                                  00007310
      SSEL(I1)=SSEL(I1)+SS(I1,J1)*DSPEL(J1)                             00007320
  200 SNEL(I1)=SNEL(I1)+SN(I1,J1)*DSPEL(J1)                             00007330
      IF(NMAT.EQ.1) GO TO 220                                           00007340
      WRITE(6,4000) I                                                   00007350
      ILO=0                                                             00007360
      IUP=0                                                             00007370
```

```
      DO 210 J=1,NPEL                                                          00007380
      IJT=NCON(I,J)                                                            00007390
      ILO=IUP+1                                                                00007400
      IUP=ILO+2                                                                00007410
  210 WRITE(6,5000) IJT,(SSEL(K),K=ILO,IUP),(SNEL(L),L=ILO,IUP)                00007420
  220 CONTINUE                                                                 00007430
 5000 FORMAT(5X,I5,6G20.13)                                                    00007440
 4000 FORMAT(//,2X,' S-T-R-E-S-S-E-S  /  S-T-R-A-I-N-S  FOR  ELEMENT',I3      00007450
     1,       //,5X,' JOINT',5X,' SSX ',17X,' SSY ',17X,'SSXY',17X,'SNX'     00007460
     2,17X,' SNY ',17X,' SNXY ',//)                                           00007470
      ICT=0                                                                    00007480
      DO 300 I2=1,NPEL                                                         00007490
      I2A=NCON(I,I2)                                                           00007500
      DO 250 J2=1,3                                                            00007510
      ICT = ICT + 1                                                            00007520
      SSJNT(I2A,J2)=SSEL(ICT)+SSJNT(I2A,J2)                                    00007530
  250 SNJNT(I2A,J2)=SNEL(ICT)+SNJNT(I2A,J2)                                    00007540
  300 SNJNT(I2A,7)=SNJNT(I2A,7)+1.                                             00007550
      DO 500 I3=1,NJT                                                          00007560
      DO 400 J3=1,3                                                            00007570
      SSJNT(I3,J3)=SSJNT(I3,J3)/SNJNT(I3,7)                                    00007580
  400 SNJNT(I3,J3)=SNJNT(I3,J3)/SNJNT(I3,7)                                    00007590
      GAU=SSJNT(I3,3)                                                          00007600
      SXPSY=(SSJNT(I3,1)+SSJNT(I3,2))/2.                                       00007610
      SXMSY=(SSJNT(I3,1)-SSJNT(I3,2))/2.                                       00007620
      AINT=DSQRT(SXMSY*SXMSY+GAU*GAU)                                          00007630
      SSJNT(I3,4)=SXPSY+AINT                                                   00007640
      SSJNT(I3,5)=SXPSY-AINT                                                   00007650
      SSJNT(I3,6)=(SSJNT(I3,4)-SSJNT(I3,5))/2.0D0                              00007660
      IF(DABS(SXMSY).LE.1.0D-6) GO TO 420                                      00007670
      RAN=-GAU/SXMSY                                                           00007680
      IF(DABS(RAN).LE.1.0D-2) GO TO 450                                        00007690
      RAN=DATAN(RAN)                                                           00007700
      GO TO 450                                                                00007710
  420 RAN=DSIGN(PI,GAU)/2.0D0                                                  00007720
  450 SSJNT(I3,7)=RAN*180.0D0/(2.0D0*PI)                                       00007730
      GAU=SNJNT(I3,3)                                                          00007740
      SXPSY=(SNJNT(I3,1)+SNJNT(I3,2))/2.0D0                                    00007750
      SXMSY=(SNJNT(I3,1)-SNJNT(I3,2))/2.0D0                                    00007760
      AINT=DSQRT(SXMSY*SXMSY+GAU*GAU)                                          00007770
      SNJNT(I3,4)=SXPSY+AINT                                                   00007780
      SNJNT(I3,5)=SXPSY-AINT                                                   00007790
      SNJNT(I3,6)=(SNJNT(I3,4)-SNJNT(I3,5))/2.0D0                              00007800
  500 SNJNT(I3,7)=SSJNT(I3,7)                                                  00007810
      WRITE(6,1000)                                                            00007820
 1000 FORMAT(///,1X,' A-V-E-R-A-G-E  S-T-R-E-S-S-E-S  AT  THE  JOINTS',       00007830
     1//,67X,'PRINCIPLE STRESSES',11X,'MAX. SHEAR',8X,'PRINCIPLE AXIS ',      00007840
     1       //,2X,' JOINT',4X,' SSX  ',12X,'SSY',15X,'SSXY',12X,'SSXM'      00007850
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     1,12X,'SSYM',12X,'SSXYM',15X,'TET(DEG.)',//)                                00007860
      DO 600 I=1,NJT                                                             00007870
  600 WRITE(6,2000) I,(SSJNT(I,J),J=1,7)                                         00007880
      WRITE(6,3000)                                                              00007890
 3000 FORMAT(///,1X,' A-V-E-R-A-G-E  S-T-R-A-I-N-S  AT  THE  JOINTS',            00007900
     1//,67X,'PRINCIPLE  STRAINS',11X,'MAX. SHEAR',8X,'PRINCIPLE AXIS ',        00007910
     1        //,2X,' JOINT',4X,' SSX  ',12X,'SSY',15X,'SSXY',12X,'SSXM'       00007920
     1,12X,'SSYM',12X,'SSXYM',15X,'TET(DEG.)',//)                               00007930
      DO 700 I=1,NJT                                                             00007940
  700 WRITE(6,2000) I,(SNJNT(I,J),J=1,6),SSJNT(I,7)                              00007950
 2000 FORMAT(2X,I4,6G17.10,2X,1G19.10)                                           00007960
      RETURN                                                                     00007970
      END                                                                        00007980

      SUBROUTINE BCOND(NBAND)                                                    00007990
      IMPLICIT REAL*8 (A-H,O-Z)                                                  00008000
      COMMON/FLPL/CJARD(156,2),CLOAD(156,2),ELCON(10,2),TITLE(20),T              00008010
      COMMON/SOL/BGK(312,64),ALOAD(312),DISP(312),ABGN                           00008020
      COMMON/INT/NEL,NJT,NMAT,NCLOAD,NPBC,NCON(100,14),NBC(156,2),NSTRES         00008030
     1,NGP,LM(22),LJT(22)                                                        00008040
      PI=3.14159265358979 3D0                                                    00008050
      NEQ=NJT*2                                                                  00008060
      DO 200 I=1,NPBC                                                            00008070
      IJT=NBC(I,1)                                                               00008080
      IEQ=IJT*2-1                                                                00008090
      KODE=NBC(I,2)                                                              00008100
      IDIA=0                                                                     00008110
      GO TO (110,120,130,140,150,160,170),KODE                                   00008120
  110 IEQ=IEQ+1                                                                  00008130
  112 BGK(IEQ,1)=ABGN                                                            00008140
      IF(IDIA.EQ.1) GO TO 132                                                    00008150
      GO TO 200                                                                  00008160
  120 GO TO 112                                                                  00008170
  130 IDIA=1                                                                     00008180
      GO TO 112                                                                  00008190
  132 IDIA=0                                                                     00008200
      GO TO 110                                                                  00008210
  140 DXY=CLOAD(IJT,2)                                                           00008220
  142 IEQ=IEQ+1                                                                  00008230
  143 ALOAD(IEQ)=ALOAD(IEQ)-BGK(IEQ,1)*DXY                                       00008240
      BGK(IEQ,1)=ABGN                                                            00008250
      DO 146 J=2,NBAND                                                           00008260
      IROW=IEQ-J+1                                                               00008270
      ICOL=IEQ-1+J                                                               00008280
      IF(IROW.LT.1) GO TO 144                                                    00008290
      ALOAD(IROW)=ALOAD(IROW)-BGK(IROW,J)*DXY                                    00008300
  144 IF(ICOL.GT.NEQ) GO TO 146                                                  00008310
```

```
      ALOAD(ICOL)=ALOAD(ICOL)-BGK(IEQ,J)*DXY                                    00008320
  146 CONTINUE                                                                  00008330
      IF(IDIA.EQ.1) GO TO 162                                                   00008340
      GO TO 200                                                                 00008350
  150 DXY=CLOAD(IJT,1)                                                          00008360
      GO TO 143                                                                 00008370
  160 IDIA=1                                                                    00008380
      DXY=CLOAD(IJT,1)                                                          00008390
      GO TO 143                                                                 00008400
  162 IDIA=0                                                                    00008410
      GO TO 140                                                                 00008420
  170 ALPHA=CLOAD(IJT,2)*PI/180.0D0                                             00008430
      CAL=DCOS(ALPHA)                                                           00008440
      SAL=DSIN(ALPHA)                                                           00008450
      IEQP=IEQ+1                                                                00008460
      DO 180 J=3,NBAND                                                          00008470
      IROW=IEQ-J+2                                                              00008480
      JM=J-1                                                                    00008490
      IF(IROW.LT.1) GO TO 175                                                   00008500
      A1=BGK(IROW,JM)*CAL+BGK(IROW,J)*SAL                                       00008510
      A2=BGK(IROW,J)*CAL-BGK(IROW,JM)*SAL                                       00008520
      BGK(IROW,JM)=A1                                                           00008530
      BGK(IROW,J )=A2                                                           00008540
  175 IF((IEQ+J-1).GT.NEQ) GO TO 180                                            00008550
      A3=BGK(IEQ,J)*CAL+BGK(IEQP,JM)*SAL                                        00008560
      A4=BGK(IEQP,JM)*CAL-BGK(IEQ,J)*SAL                                        00008570
      BGK(IEQ ,J )=A3                                                           00008580
      BGK(IEQP,JM)=A4                                                           00008590
  180 CONTINUE                                                                  00008600
      A1=BGK(IEQ,1)*CAL*CAL+BGK(IEQ,2)*CAL*SAL*2.0D0+BGK(IEQP,1)*SAL*SAL        00008610
      A2=BGK(IEQ,2)*(CAL*CAL-SAL*SAL)+(BGK(IEQP,1)-BGK(IEQ,1))*SAL*CAL          00008620
      BGK(IEQ,1)=A1                                                             00008630
      BGK(IEQ,2)=A2                                                             00008640
      BGK(IEQP,1)=ABGN                                                          00008650
  200 CONTINUE                                                                  00008660
      RETURN                                                                    00008670
      END                                                                       00008680

      SUBROUTINE DISPL                                                          00008690
      IMPLICIT REAL*8 (A-H,O-Z)                                                 00008700
      COMMON/FLPL/COARD(156,2),CLOAD(156,2),ELCON(10,2),TITLE(20),T             00008710
      COMMON/SOL/BGK(312,64),ALOAD(312),DISP(312),ABGN                          00008720
      COMMON/INT/NEL,NJT,NMAT,NCLOAD,NPBC,NCON(100,14),NBC(156,2),NSTRES        00008730
     1,NGP,LM(22),LJT(22)                                                       00008740
      DIMENSION REACT(157,2)                                                    00008750
      EQUIVALENCE (BGK(1,1),REACT(1,1))                                         00008760
      PI=3.141592653589793D0                                                    00008770
```

```
      DO 200 I=1,NPBC                                                   00008780
      IJT=NBC(I,1)                                                      00008790
      IEQ=IJT*2-1                                                       00008800
      KODE=NBC(I,2)                                                     00008810
      IDIA=0                                                            00008820
      DPBC=0.0D0                                                        00008830
      GO TO (110,120,130,140,150,160,170),KODE                          00008840
  110 IEQ=IEQ+1                                                         00008850
      JR=2                                                              00008860
  112 REACT(I,JR)=-DISP(IEQ)*ABGN                                       00008870
      DISP(IEQ)=DPBC                                                    00008880
      IF(IDIA.EQ.1) GO TO 132                                           00008890
      IF(IDIA.EQ.2) GO TO 162                                           00008900
      GO TO 200                                                         00008910
  120 JR=1                                                              00008920
      GO TO 112                                                         00008930
  130 IDIA=1                                                            00008940
      JR=1                                                              00008950
      GO TO 112                                                         00008960
  132 IDIA=0                                                            00008970
      GO TO 110                                                         00008980
  140 DPBC=CLOAD(IJT,2)                                                 00008990
      GO TO 110                                                         00009000
  150 JR=1                                                              00009010
      DPBC=CLOAD(IJT,1)                                                 00009020
      GO TO 112                                                         00009030
  160 IDIA=2                                                            00009040
      JR=1                                                              00009050
      DPBC=CLOAD(IJT,1)                                                 00009060
      GO TO 112                                                         00009070
  162 IDIA=0                                                            00009080
      DPBC=CLOAD(IJT,2)                                                 00009090
      GO TO 110                                                         00009100
  170 ALPHA=CLOAD(IJT,2)*PI/180.0D0                                     00009110
      CAL=DCOS(ALPHA)                                                   00009120
      SAL=DSIN(ALPHA)                                                   00009130
      A1=DISP(IEQ)*CAL                                                  00009140
      A2=DISP(IEQ)*SAL                                                  00009150
      IEQP=IEQ+1                                                        00009160
      A3=-DISP(IEQP)*SAL*ABGN                                           00009170
      A4= DISP(IEQP)*CAL*ABGN                                           00009180
      DISP(IEQ)=A1                                                      00009190
      DISP(IEQP)=A2                                                     00009200
      REACT(I   ,1)=-A3                                                 00009210
      REACT(I   ,2)=-A4                                                 00009220
  200 CONTINUE                                                          00009230
      REACT(157,1)=0.0D0                                                00009240
      REACT(157,2)=0.0D0                                                00009250
```

```
      DO 300 I=1,NP3C                                                   00009260
      REACT(157,1)=REACT(157,1)+REACT(I,1)                              00009270
  300 REACT(157,2)=REACT(157,2)+REACT(I,2)                              00009280
      RETURN                                                            00009290
      END                                                               00009300

      SUBROUTINE PUNCH                                                  00009310
C     ****************************************************************00009320
C                                                                       00009330
C     SUBROUTINE PUNCH PRODUCES DISPLACEMENT OUTPUT ON PUNCHED CARDS    00009340
C     TO BE USED AS DATA INPUT TO PROGRAM ELASTIC                       00009350
C                                                                       00009360
C     ****************************************************************00009370
C                                                                       00009380
      IMPLICIT REAL*8 (A-H,O-Z)                                         00009390
      COMMON/SOL/BG(312,64),ALOAD(312),DISP(312),ABGN                   00009400
      COMMON/INT/NEL,NJT,NMAT,NCLOAD,NPBC,NCON(100,14),NBC(156,2),NSTRES00009410
     1,NGP,LM(22),LJT(22)                                               00009420
      DO 20 I=1,NJT                                                     00009430
      II=I*2-1                                                          00009440
      III=II+1                                                          00009450
      WRITE(7,10) I,DISP(II),DISP(III)                                  00009460
   10 FORMAT(1I10,2G24.16)                                              00009470
   20 CONTINUE                                                          00009480
      RETURN                                                            00009490
      END                                                               00009500
```

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LIEUTENANT, U. S. NAVY

CHARACTERIZATION OF ELASTIC SOLIDS USING FINITE ELEMENT METHODS

U. S. NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA
DECEMBER 1972

```
C     ************************************************************00000010
C                                                                  00000020
C     PROGRAM DLASTIC DETERMINES THE SIX ELASTIC CONSTANTS <C11,C12,C13,00000030
C     C22,C23,C33>BY DIRECT INVERSION OF THE FINITE ELEMENT FORMULATION 00000040
C     IN ORDER TO CHARACTERIZE A GENERAL (LINEAR) ANISOTROPIC MATERIAL. 00000050
C     DISPLACEMENTS AND FORCES AT ALL NODAL POINTS ARE KNOWN.           00000060
C                                                                       00000070
C     IMPLICIT REAL*8(A-H,O-Z)                                          00000080
C                                                                       00000090
C     ELEMENT GEOMETRY USED IN DLASTIC IS PRESENTLY RESTRICTED TO THE   00000100
C     FOLLOWING 4-NODED, RECTANGULAR SHAPE:                             00000110
C                                                                       00000120
C                                J .-------------. I                    00000130
C                                  |        V|___|                     00000140
C                                B |         |___U                     00000150
C                                  |         |                         00000160
C                                K .-------------. L                    00000170
C                                          A                            00000180
C                                                                       00000190
C     THE HORIZONTAL DIMENSION, A, IS CALCULATED IN DLASTIC AS THE AVER-00000200
C     AGE OF TOP ELEMENT AND BOTTOM ELEMENT HORIZ. DIMENSIONS.          00000210
C     THE VERTICAL DIMENSION, B, IS CALCULATED AS THE AVERAGE OF RIGHT  00000220
C     AND LEFT VERTICAL SIDE DIMENSIONS.                                00000230
C                                                                       00000240
C     ************************************************************00000250
      COMMON/FLP/BK(200,6),BKT(6,200),R(200),COORD(100,2),SS(100,6),SN( 00000260
     1100,6),SK(8,6),BSYM(6,6),ASYM(6,6),U(100),V(100),BR(6),C(6),A,B,  00000270
     1TITLE(20),T                                                       00000280
C     ************************************************************00000290
C                                                                       00000300
C     INPUT CARDS REQUIRED:                                             00000310
C      TITLE (20A4)                                                     00000320
C      PARAMETERS (2I10,10X,1G24.16)                                    00000330
C        COL 1-10 NUMBER OF ELEMENTS IN STRUCTURE (NEL)  MAX. NEL=50    00000340
C           11-20                 JOINTS             (NJT)  MAX. NJT=100 00000350
C           31-55 THICKNESS, T.   T=0.0 OR BLANK SAME AS UNIT THICK T=1.00000360
C                                                                       00000370
```

```
C       ELEMENTAL CONNECTIVITY (1-CARD PER ELEMENT)                          00000380
C         COL 1-10 ELEMENT ID, NEID (1I10)                                   00000390
C             11-15  JOINT ID, JOINT I OF ELEMENT                            00000400
C             16-20  JOINT ID, JOINT J OF ELEMENT                            00000410
C             21-25  JOINT ID, JOINT K OF ELEMENT                            00000420
C             26-30  JOINT ID, JOINT L OF ELEMENT                            00000430
C                                                                            00000440
      COMMON/INT/NCON(50,4),L1(6),L2(6),NEL,NJT,NEQ,NEID(50),JID(100),IO     00000450
C                                                                            00000460
C     JOINT COORDINATES (1-CARD PER JOINT)                                   00000470
C      COL 1-10 JOINT ID, JID (1I10)                                         00000480
C            11-34 X-COORDINATE OF JOINT (1G24.16)                           00000490
C            35-58 Y-COORDINATE OF JOINT (1G24.16)                           00000500
C     JOINT FORCES (1-CARD PER JOINT)                                        00000510
C      COL 1-10 JOINT ID, JID (1I10)                                         00000520
C            11-34 RESULTANT HORIZONTAL FORCE ACTING ON JOINT (1G24.16)      00000530
C            35-58 RESULTANT VERTICAL   FORCE ACTING ON JOINT (1G24.16)      00000540
C     JOINT DISPLACEMENTS (1-CARD PER JOINT)                                 00000550
C      COL 1-10 JOINT ID, JID (1I10)                                         00000560
C            11-34 HORIZONTAL DISPLACEMENT OF JOINT (1G24.16)                00000570
C            35-58 VERTICAL   DISPLACEMENT OF JOINT (1G24.16)                00000580
C                                                                            00000590
C               CODED BY BRIAN A. EDWARDS, LT., USN (AUG. 72)                00000600
C     ***************************************************************      00000610
   99 CALL INPUT                                                             00000620
      CALL MERGE                                                             00000630
      CALL BMULT                                                             00000640
      CALL SINGL                                                             00000650
      CALL INVRT(BSYM,6,B,L1,L2)                                             00000660
      CALL ANSER                                                             00000670
      GO TO 99                                                               00000680
      END                                                                    00000690

      SUBROUTINE INPUT                                                       00000700
      IMPLICIT REAL*8(A-H,O-Z)                                               00000710
      COMMON/INT/NCON(50,4),L1(6),L2(6),NEL,NJT,NEQ,NEID(50),JID(100),IO     00000720
      COMMON/FLP/BK(200,6),BKT(6,200),R(200),COORD(100,2),SS(100,6),SN(      00000730
     1100,6),SK(8,6),BSYM(6,6),ASYM(6,6),U(100),V(100),BR(6),C(6),A,B,       00000740
     1TITLE(20),T                                                            00000750
      DATA CHK/'STOP'/                                                       00000760
      READ (5,10) TITLE                                                      00000770
   10 FORMAT (20A4)                                                          00000780
      IF (TITLE(1).EQ.CHK) STOP                                              00000790
      WRITE (6,11) TITLE                                                     00000800
   11 FORMAT ('1',20A4,//)                                                   00000810
C     READ PARAMETERS                                                        00000820
      READ (5,15) NEL,NJT,T                                                  00000830
```

```
   15 FORMAT (2I10,10X,1G24.16)                                          00000840
      IF (T.GT.0.0) GO TO 1150                                           00000850
      T=1.0D0                                                            00000860
 1150 NEQ=NJT*2                                                          00000870
      WRITE (6,16) NEL,NJT,T                                             00000880
   16 FORMAT('0','NUMBER OF ELEMENTS IN STRUCTURE',1I10,                 00000890
     1/,' ','NUMBER OF JOINTS IN STRUCTURE',1I10,                        00000900
     1/,' ','THICKNESS IS(UNIFORM)=',1G24.16)                            00000910
C     READ CONNECTIVITY MATRIX                                           00000920
      DO 20 I=1,NEL                                                      00000930
   20 READ (5,25) NEID(I), (NCON(I,J),J=1,4)                             00000940
   25 FORMAT (1I10,4I5)                                                  00000950
      WRITE (6,30)                                                       00000960
   30 FORMAT('0','CONNECTIVITY MATRIX',//,6X,'EL.ID',4X,'I-J-K-L CONNECT00000970
     1IVITY')                                                            00000980
      DO 35 I=1,NEL                                                      00000990
   35 WRITE(6,36) NEID(I), (NCON(I,J),J=1,4)                             00001000
   36 FORMAT (' ',1I10,4I5)                                              00001010
C     READ COORDINATES OF EACH JOINT                                     00001020
      WRITE (6,46)                                                       00001030
   46 FORMAT('0',2X,'JOINT ID',12X,'X-COORDINATE',12X,'Y-COORDINATE')    00001040
      DO 40 M=1,NJT                                                      00001050
      DO 40 N=1,2                                                        00001060
   40 COORD(M,N)=0.0D0                                                   00001070
      DO 42 M=1,NJT                                                      00001080
      READ(5,45) JID(M), (COORD(M,N),N=1,2)                              00001090
   45 FORMAT(1I10,2G24.16)                                               00001100
   42 WRITE(6,44) JID(M), (COORD(M,N),N=1,2)                             00001110
   44 FORMAT(' ',1I10,2G24.16)                                           00001120
C     READ FORCES ACTING AT EACH JOINT                                   00001130
      WRITE (6,56)                                                       00001140
   56 FORMAT('0',2X,'         ',12X,'HORIZ. FORCE',12X,' VERT. FORCE')   00001150
      DO 50 M=1,NEQ                                                      00001160
   50 R(M)=0.0D0                                                         00001170
      READ (5,55)      (R(M),M=1,NEQ)                                    00001180
   55 FORMAT ( 10X,2G24.16)                                              00001190
   52 WRITE(6,54)     (R(M),M=1,NEQ)                                     00001200
   54 FORMAT (' ', 10X,2G24.16)                                          00001210
C     READ DISPLACEMENTS OF EACH JOINT                                   00001220
      WRITE(6,66)                                                        00001230
   66 FORMAT('0',2X,'JOINT ID',10X,'X-DISPLACEMENT',10X,'Y-DISPLACEMENT'00001240
     1)                                                                  00001250
      DO 60 M=1,NJT                                                      00001260
      U(M)=0.0D0                                                         00001270
   60 V(M)=0.0D0                                                         00001280
      DO 62 M=1,NJT                                                      00001290
      READ (5,65) JID(M),U(M),V(M)                                       00001300
   65 FORMAT(1I10,2G24.16)                                               00001310
```

```
   62 WRITE(6,64) JID(M),U(M),V(M)                                          00001320
   64 FORMAT(' ',1I10,2G24.16)                                              00001330
      RETURN                                                                00001340
      END                                                                   00001350

      SUBROUTINE MERGE                                                      00001360
      IMPLICIT REAL*8(A-H,O-Z)                                              00001370
      COMMON/INT/NCON(50,4),L1(6),L2(6),NEL,NJT,NEQ,NEID(50),JID(100),IO000001380
      COMMON/FLP/BK(200,6),BKT(6,200),R(200),COORD(100,2),SS(100,6),SN( 00001390
     1100,6),SK(8,6),BSYM(6,6),ASYM(6,6),U(100),V(100),BR(6),C(6),A,B,  00001400
     1TITLE(20),T                                                       00001410
C     ***********************************************************************00001420
C     SUBROUTINE MERGE FORMS THE MASTER MODIFIED STIFFNESS MATRIX       00001430
C        BK---MASTER MODIFIED STIFFNESS MATRIX                          00001440
C        SK---ELEMENT MODIFIED STIFFNESS MATRIX                         00001450
C        NCON-CONNECTIVITY MATRIX                                       00001460
C        NEL--NUMBER OF ELEMENTS                                        00001470
C        NEQ--NUMBER OF EQUATIONS, NJT*2                                00001480
C        NJT--NUMBER OF JOINTS IN MASTER STRUCTURE                      00001490
C        U,V--HORIZ. AND VERT. DISPLACEMENTS AT A JOINT                 00001500
C        T----THICKNESS (ASSUMED UNIFORM)                               00001510
C     ***********************************************************************00001520
      DO 100 I=1,NEQ                                                    00001530
      DO 100 J=1,6                                                      00001540
  100 BK(I,J)=0.0D0                                                     00001550
C     DETERMINE EACH ELEMENTS MODIFIED STIFFNESS MATRIX                 00001560
      DO 150 I=1,NJT                                                    00001570
      DO 150 J=1,6                                                      00001580
      SS(I,J)=0.0D0                                                     00001590
  150 SN(I,J)=0.0D0                                                     00001600
      DO 250 IO=1,NEL                                                   00001610
      CALL STIFF                                                        00001620
      DO 200 M=1,4                                                      00001630
      IJ=NCON(IO,M)                                                     00001640
      II=(M*2)-1                                                        00001650
      JJ=II+1                                                           00001660
      DO 175 K=1,6                                                      00001670
      SS(IJ,K)=SS(IJ,K)+SK(II,K)                                        00001680
      SN(IJ,K)=SN(IJ,K)+SK(JJ,K)                                        00001690
  175 CONTINUE                                                          00001700
  200 CONTINUE                                                          00001710
  250 CONTINUE                                                          00001720
      DO 300 I=1,NJT                                                    00001730
      I1=(I*2)-1                                                        00001740
      I2=I1+1                                                           00001750
      DO 275 K=1,6                                                      00001760
      BK(I1,K)=SS(I,K)*T                                                00001770
```

```
      BK(I2,K)=SN(I,K)*T                                                    00001780
  275 CONTINUE                                                              00001790
  300 CONTINUE                                                              00001800
      RETURN                                                                00001810
      END                                                                   00001820

      SUBROUTINE STIFF                                                      00001830
      IMPLICIT REAL*8(A-H,O-Z)                                              00001840
      COMMON/INT/NCON(50,4),L1(6),L2(6),NEL,NJT,NEQ,NEID(50),JID(100),IO00001850
      COMMON/FLP/BK(200,6),BKT(6,200),R(200),COORD(100,2),SS(100,6),SN( 00001860
     1100,6),SK(8,6),BSYM(6,6),ASYM(6,6),U(100),V(100),BR(6),C(6),A,B, 00001870
     1TITLE(20),T                                                           00001880
      DIMENSION SSK(10)                                                     00001890
C     ********************************************************************00001900
C     SUBROUTINE STIFF LOADS THE ELEMENTAL MODIFIED STIFFNESS MATRIX    00001910
C        SK---ELEMENT MODIFIED STIFFNESS MATRIX                         00001920
C        NCON-CONNECTIVITY MATRIX                                       00001930
C        A,B--HORIZ. AND VERT. DIMENSIONS OF AN ELEMENT                 00001940
C        U,V--HORIZ. AND VERT. DISPLACEMENTS AT A JOINT                 00001950
C        NEL--NUMBER OF ELEMENTS                                        00001960
C        NJT--NUMBER OF JOINTS IN MASTER STRUCTURE                      00001970
C     ********************************************************************00001980
      DO 10 I=1,10                                                          00001990
   10 SSK(I)=0.0D0                                                          00002000
      DO 20 I=1,8                                                           00002010
      DO 20 J=1,6                                                           00002020
   20 SK(I,J)=0.0D0                                                         00002030
C     CALCULATE THE HORIZONTAL DIMENSION, A                             00002040
      A1=COORD(NCON(IO,1),1)                                            00002050
      A2=COORD(NCON(IO,2),1)                                            00002060
      DA=DABS(A1-A2)                                                    00002070
      A1=COORD(NCON(IO,4),1)                                            00002080
      A2=COORD(NCON(IO,3),1)                                            00002090
      DB=DABS(A1-A2)                                                    00002100
      A=(DA+DB)/2.0D0                                                   00002110
C     CALCULATE THE VERTICAL DIMENSION, B                               00002120
      A1=COORD(NCON(IO,4),2)                                            00002130
      A2=COORD(NCON(IO,1),2)                                            00002140
      DA=DABS(A1-A2)                                                    00002150
      A1=COORD(NCON(IO,3),2)                                            00002160
      A2=COORD(NCON(IO,2),2)                                            00002170
      DB=DABS(A1-A2)                                                    00002180
      B=(DA+DB)/2.0D0                                                   00002190
C     DEFINE SOME COMMON MULTIPLICATIVE CONSTANTS                       00002200
      D1=1.0/2.0D0                                                      00002210
      D2=1.0/3.0D0                                                      00002220
      D3=1.0/4.0D0                                                      00002230
```

```
      D4=A/B                                                                    00002240
      D5=B/A                                                                    00002250
C     DEFINE THE TERMS OF AUXILIARY ARRAY SSK                                   00002260
      SSK(1)=D1*(U(NCON(IO,1))-U(NCON(IO,3)))                                   00002270
      SSK(2)=D2*D5*(V(NCON(IO,1))-V(NCON(IO,2))-D1*V(NCON(IO,3))+D1*V(NC00002280
     1ON(IO,4)))                                                                00002290
      SSK(3)=D1*(-U(NCON(IO,2))+U(NCON(IO,4)))                                  00002300
      SSK(4)=D2*D5*(-D1*V(NCON(IO,1))+D1*V(NCON(IO,2))+V(NCON(IO,3))-V(N00002310
     1CON(IO,4)))                                                               00002320
      SSK(5)=D1*(V(NCON(IO,1))-V(NCON(IO,3)))                                   00002330
      SSK(6)=D2*D4*(U(NCON(IO,1))+D1*U(NCON(IO,2))-D1*U(NCON(IO,3))-U(NC00002340
     1CN(IO,4)))                                                                00002350
      SSK(7)=D1*(-V(NCON(IO,2))+V(NCON(IO,4)))                                  00002360
      SSK(8)=D2*D4*(D1*U(NCON(IO,1))+U(NCON(IO,2))-U(NCON(IO,3))-D1*U(NC00002370
     1ON(IO,4)))                                                                00002380
      SSK(9)=D3*(V(NCON(IO,1))-V(NCON(IO,2))-V(NCON(IO,3))+V(NCON(IO,4))00002390
     1)                                                                         00002400
      SSK(10)=D3*(U(NCON(IO,1))+U(NCON(IO,2))-U(NCON(IO,3))-U(NCON(IO,4)00002410
     1))                                                                        00002420
C     LOAD THE ELEMENTAL MODIFIED STIFFNESS MATRIX                              00002430
      SK(1,1)=D2*D5*(U(NCON(IO,1))-U(NCON(IO,2))-D1*U(NCON(IO,3))+D1*U(N00002440
     1CON(IO,4)))                                                               00002450
      SK(1,2)=D3*(V(NCON(IO,1))+V(NCON(IO,2))-V(NCON(IO,3))-V(NCON(IO,4)00002460
     1))                                                                        00002470
      SK(3,1)=-SK(1,1)                                                          00002480
      SK(5,1)=D2*D5*(-D1*U(NCON(IO,1))+D1*U(NCON(IO,2))+U(NCON(IO,3))-U(00002490
     1NCON(IO,4)))                                                              00002500
      SK(7,1)=-SK(5,1)                                                          00002510
      SK(6,6)=-SSK(10)+SSK(4)                                                   00002520
      SK(3,6)=SSK(9)+SSK(8)                                                     00002530
      SK(2,6)=SSK(10)+SSK(2)                                                    00002540
      SK(1,6)=SSK(9)+SSK(6)                                                     00002550
      SK(8,5)=-(SSK(7)+SSK(6))                                                  00002560
      SK(6,5)=-(SSK(5)+SSK(8))                                                  00002570
      SK(4,5)=SSK(7)+SSK(8)                                                     00002580
      SK(2,5)=SSK(5)+SSK(6)                                                     00002590
      SK(4,4)=D2*D4*(D1*V(NCON(IO,1))+V(NCON(IO,2))-V(NCON(IO,3))-D1*V(N00002600
     1CON(IO,4)))                                                               00002610
      SK(2,4)=D2*D4*(V(NCON(IO,1))+D1*V(NCON(IO,2))-D1*V(NCON(IO,3))-V(N00002620
     1CON(IO,4)))                                                               00002630
      SK(7,3)=-(SSK(3)+SSK(4))                                                  00002640
      SK(5,3)=-SSK(1)+SSK(4)                                                    00002650
      SK(3,3)=SSK(3)-SSK(2)                                                     00002660
      SK(1,3)=SSK(1)+SSK(2)                                                     00002670
      SK(2,2)=D3*(U(NCON(IO,1))-U(NCON(IO,2))-U(NCON(IO,3))+U(NCON(IO,4)00002680
     1))                                                                        00002690
      SK(3,2)=-SK(1,2)                                                          00002700
      SK(4,2)=SK(2,2)                                                           00002710
```

```
      SK(5,2)=-SK(1,2)                                                  00002720
      SK(6,2)=-SK(2,2)                                                  00002730
      SK(7,2)=SK(1,2)                                                   00002740
      SK(8,2)=SK(6,2)                                                   00002750
      SK(2,3)=SK(1,1)                                                   00002760
      SK(4,3)=-SK(1,1)                                                  00002770
      SK(6,3)=SK(5,1)                                                   00002780
      SK(8,3)=-SK(5,1)                                                  00002790
      SK(6,4)=-SK(4,4)                                                  00002800
      SK(8,4)=-SK(2,4)                                                  00002810
      SK(1,5)=SK(2,4)                                                   00002820
      SK(3,5)=SK(4,4)                                                   00002830
      SK(5,5)=SK(6,4)                                                   00002840
      SK(7,5)=SK(8,4)                                                   00002850
      SK(4,6)=-SK(2,6)                                                  00002860
      SK(5,6)=-SK(3,6)                                                  00002870
      SK(7,6)=-SK(1,6)                                                  00002880
      SK(8,6)=-SK(6,6)                                                  00002890
      RETURN                                                            00002900
      END                                                               00002910

      SUBROUTINE BMULT                                                  00002920
      IMPLICIT REAL*8(A-H,O-Z)                                          00002930
      COMMON/INT/NCON(50,4),L1(6),L2(6),NEL,NJT,NEQ,NEID(50),JID(100),IO00002940
C     ****************************************************************00002950
C     SUBROUTINE BMULT PRE-MULTIPLIES THE MASTER STIFFNESS MATRIX AND   00002960
C     THE FORCE VECTOR BY THE TRANSPOSE OF BK                           00002970
C        BK---MASTER MODIFIED STIFFNESS MATRIX                          00002980
C        BKT--TRANSPOSE OF THE MASTER STIFFNESS MATRIX                  00002990
C        BSYM-SYMMETRIC SQUARE MATRIX,BKT*BK                            00003000
C        NEQ--NUMBER OF EQUATIONS,NJT*2                                 00003010
C        R----FORCE VECTOR                                              00003020
C        BR---COLUMN VECTOR,BKT*R                                       00003030
C     ****************************************************************00003040
      COMMON/FLP/BK(200,6),BKT(6,200),R(200),COORD(100,2),SS(100,6),SN( 00003050
     1100,6),SK(8,6),BSYM(6,6),ASYM(6,6),U(100),V(100),BR(6),C(6),A,B,  00003060
     1TITLE(20),T                                                       00003070
C     FORM THE TRANSPOSE OF THE MASTER MODIFIED STIFFNESS MATRIX        00003080
      DO 10 I=1,6                                                       00003090
      DO 10 J=1,NEQ                                                     00003100
   10 BKT(I,J)=0.0D0                                                    00003110
      DO 20 I=1,6                                                       00003120
      DO 20 J=1,NEQ                                                     00003130
   20 BKT(I,J)=BK(J,I)                                                  00003140
C     FORM THE PRODUCT, BSYM=BKT*BK                                     00003150
      DO 30 I=1,6                                                       00003160
      DO 30 J=1,6                                                       00003170
```

```
      BSYM(I,J)=0.0D0                                                   00003180
      DO 30 K=1,NEQ                                                     00003190
   30 BSYM(I,J)=BSYM(I,J)+BKT(I,K)*BK(K,J)                              00003200
C     FORM THE PRODUCT, BR=BKT*R                                        00003210
      DO 40 I=1,6                                                       00003220
      BR(I)=0.0D0                                                       00003230
      DO 40 K=1,NEQ                                                     00003240
   40 BR(I)=BR(I)+BKT(I,K)*R(K)                                         00003250
      RETURN                                                            00003260
      END                                                               00003270

      SUBROUTINE SINGL                                                  00003280
      IMPLICIT REAL*8(A-H,O-Z)                                          00003290
      COMMON/INT/NCON(50,4),L1(6),L2(6),NEL,NJT,NEQ,NEID(50),JID(100),IO00003300
      COMMON/FLP/BK(200,6),BKT(6,200),R(200),COORD(100,2),SS(100,6),SN( 00003310
     1100,6),SK(8,6),BSYM(6,6),ASYM(6,6),U(100),V(100),BR(6),C(6),A,B,  00003320
     1TITLE(20),T                                                       00003330
C     ****************************************************************00003340
C                                                                       00003350
C     SUBROUTINE SINGL CALCULATES THE EIGEN VALUES ASSOCIATED WITH BSYM.00003360
C     A ZERO EIGEN VALUE INDICATES BSYM TO BE SINGULAR.                 00003370
C                                                                       00003380
C     ****************************************************************00003390
      DO 10 I=1,6                                                       00003400
      DO 10 J=1,6                                                       00003410
   10 ASYM(I,J)=0.0D0                                                   00003420
      DO 15 I=1,6                                                       00003430
      DO 15 J=1,6                                                       00003440
   15 ASYM(I,J)=BSYM(I,J)                                               00003450
      CALL EIGEN (ASYM,6,0,C,A,6)                                       00003460
      WRITE(6,20)                                                       00003470
   20 FORMAT('0','EIGEN VALUES OF BSYM=BKT*BK')                         00003480
      DO 30 I=1,6                                                       00003490
   30 WRITE (6,35) C(I)                                                 00003500
   35 FORMAT(' ',10X,1G24.16)                                           00003510
      RETURN                                                            00003520
      END                                                               00003530

      SUBROUTINE EIGEN  (A,N,NOYES,EIVU,EIVR,NDIM)                      00003540
C.....................................................................00003550
C                                                                       00003560
C     SUGGESTED FOR USE AT NPG BY PROF. J. G. CANTIN, CODE 59CI         00003570
C     ORIGINALLY PROGRAMMED BY W. POOLE, COMPUTER CENTER, U.C.BERKELEY  00003580
C                                                                       00003590
C  PURPOSE                                                              00003600
C     THIS SUBROUTINE CALCULATES IN DOUBLE PRECISION ALL THE EIGENVAL-  00003610
```

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C      UES OF A REAL, SYMMETRIC MATRIX BY USE OF THE JACOBI VARIABLE      00003620
C      THRESHOLD METHOD.                                                  00003630
C                                                                         00003640
C   USAGE                                                                 00003650
C      CALL EIGEN (A,N,NOYES,EIVU,EIVR,NDIM)                              00003660
C                                                                         00003670
C   DESCRIPTION OF PARAMETERS                                             00003680
C      A  -  THE REAL*8 SYMMETRIC (SQUARE) MATRIX WHOSE EIGENVALUES AND   00003690
C            EIGENVECTORS ARE SOUGHT. ROW-DIMENSION MUST BE NDIM.         00003700
C      N  -  ORDER OF A, 1.LE.N.LE.NDIM .LE. 160                          00003710
C      NOYES-INDICATOR SUPPLIED BY USER -                                 00003720
C                     IF NOYES = 0, EIGENVECTORS ARE NOT WANTED,          00003730
C                     IF NOYES = 1, EIGENVECTORS ARE WANTED.              00003740
C      EIVU - THE EIGENVALUES ARE RETURNED IN THIS VECTOR, WHICH MUST BE  00003750
C             DIMENSIONED TO AT LEAST N IN CALLING PROGRAM. THEY ARE IN   00003760
C             NO PARTICULAR ORDER. (REAL*8)                               00003770
C      IMPLICIT REAL*8(A-H,O-Z)                                           00003780
C      EIVR - THE EIGENVECTORS (IF REQUESTED) ARE RETURNED AS THE COLUMNS 00003790
C             OF THIS MATRIX, THE ROW-DIMENSION OF WHICH MUST BE NDIM IN  00003800
C             THE CALLING PROGRAM. THE COLUMN DIMENSION MUST BE AT LEAST  00003810
C             N IN THE CALLING PROGRAM. THE VECTOR EIVR(I,J),I=1,N        00003820
C             CORRESPONDS TO THE EIGENVALUE EIVU(J). IF EIGENVECTORS      00003830
C             ARE NOT REQUESTED, AN UNDIMENSIONED VARIABLE (E.G. DUMMY)   00003840
C             SHOULD BE PUT IN THE POSITION OF EIVR IN THE CALL STATE-    00003850
C             MENT. EIVR IS REAL*8.                                       00003860
C      NDIM - THE ROW-DIMENSION   (FIRST DIMENSION) OF A AND EIVR IN THE  00003870
C             DIMENSION STATEMENT OF THE CALLING PROGRAM. NDIM.LE.160     00003880
C                                                                         00003890
C   REMARKS                                                               00003900
C      1)  MATRIX A IS DESTROYED DURING CALCULATION.                      00003910
C      2)  ALL EIGENVALUES ARE CALCULATED EACH TIME ROUTINE IS CALLED.    00003920
C      3)  VARIOUS DIAGNOSTIC MESSAGES MAY BE SENT TO STANDARD OUTPUT     00003930
C          MEDIUM BY EIGEN . THESE INDICATE IMPROPER ARGUMENTS AND ARE    00003940
C          GENERALLY SELF-EXPLANATORY.                                    00003950
C      4)  LIMIT OF 160 FOR N (AND NDIM) COULD BE RAISED BY CHANGING      00003960
C          STATEMENT 21.                                                  00003970
C                                                                         00003980
C   METHOD                                                                00003990
C      SEE REFERENCES AND WRITEUP OF F2-BC-JCVT ON FILE AT COMPUTER       00004000
C      FACILITY,NPG.                                                      00004010
C                                                                         00004020
       DIMENSION A(NDIM,NDIM),EIVU(NDIM),EIVR(NDIM,NDIM)                  00004030
C   OBSERVATION                                                           00004040
C      TAKEN FROM P.281 OF REFERENCE 2 -                                  00004050
C                                                                         00004060
C     "EVEN WHEN (THE MATRIX) HAS SOME PATHOLOGICALLY CLOSE EIGENVALUES   00004070
C      THE CORRESPONDING EIGENVECTORS ARE ALMOST EXACTLY ORTHOGONAL,      00004080
C      THOUGH THEY MAY BE VERY INACCURATE; IT IS EVIDENT THAT THEY SPAN   00004090
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C      THE CORRECT SUBSPACE AND GIVE FULL DIGITAL INFORMATION. THIS IS    00004100
C      PERHAPS THE MOST SATISFACTORY FEATURE OF JACOBI'S METHOD. THE      00004110
C      OTHER TECHNIQUES ... ARE SUPERIOR ... IN SPEED AND ACCURACY, BUT   00004120
C      IT IS COMPARATIVELY TIRESOME TO OBTAIN ORTHOGONAL EIGENVECTORS     00004130
C      CORRESPONDING TO PATHOLOGICALLY CLOSE OR COINCIDENT EIGENVALUES. "00004140
C                                                                         00004150
C   REFERENCES                                                            00004160
C      1)  NEUMANN, J. VON AND GOLDSTEIN, "THE JACOBI METHOD FOR REAL     00004170
C          SYMMETRIC MATRICES", JOURNAL OF ACM, 1, 1959.                  00004180
C                                                                         00004190
C      2)  WILKINSON, J.H., "THE ALGEBRAIC EIGENVALUE PROBLEM", OXFORD    00004200
C          U. PRESS, 1965, P.277 FF.                                      00004210
C                                                                         00004220
C.........................................................................00004230
      IF(N-1)20,23,21                                                     00004240
   20 PRINT 22,N                                                          00004250
   22 FORMAT(4H N= ,I3,68H IS TOO SMALL.  LIMIT IS 1.  RETURN TO CALLING00004260
     1 ROUTINE FROM EIGEN . )                                             00004270
      RETURN                                                              00004280
   23 PRINT 24, A(1,1)                                                    00004290
   24 FORMAT (24H IN EIGEN  , MATRIX A = ,E14.6)                          00004300
      RETURN                                                              00004310
   21 IF(N-160)1,1,3                                                      00004320
    3 PRINT 2,N                                                           00004330
    2 FORMAT(3H N=,I5,'   IS TOO LARGE.  LIMIT IS 160.   RETURN TO CALLI00004340
     1NG ROUTINE FROM EIGEN .')                                           00004350
      RETURN                                                              00004360
    1 IF(NOYES)99,102,99                                                  00004370
   99 CONTINUE                                                            00004380
      DO 101 J=1,N                                                        00004390
      DO 100 I=1,N                                                        00004400
  100 EIVR(I,J)=0.0                                                       00004410
  101 EIVR(J,J)=1.0                                                       00004420
  102 ATOP=0.                                                             00004430
      DO 112 J=1,N                                                        00004440
      DO 111 I=1,J                                                        00004450
      IF(A(I,J)-A(J,I))90,103,90                                          00004460
   90 CONTINUE                                                            00004470
      A(I,J)=.5*(A(I,J)+A(J,I))                                           00004480
      A(J,I)=A(I,J)                                                       00004490
  103 CONTINUE                                                            00004500
      IF(ATOP-DABS(A(I,J)))104,111,111                                    00004510
  104 ATOP=DABS(A(I,J))                                                   00004520
  111 CONTINUE                                                            00004530
  112 EIVU(J)=A(J,J)                                                      00004540
      IF(ATOP)109,109,113                                                 00004550
  109 PRINT 110                                                           00004560
  110 FORMAT(26H IN EIGEN , MATRIX A  = 0 )                               00004570
```

```
      RETURN                                                            00004580
  113 AVGF=DFLOAT(N*(N-1))*.55                                          00004590
      D=0.0                                                             00004600
      DO 114 JJ=2,N                                                     00004610
      DO 114 II=2,JJ                                                    00004620
      S=A(II-1,JJ)/ATOP                                                 00004630
  114 D=S*S+D                                                           00004640
      DSTOP=(1.D-06)*D                                                  00004650
      THRSH =DSQRT (D/AVGF)*ATOP                                        00004660
  115 IFLAG=0                                                           00004670
      DO 130 JCOL=2,N                                                   00004680
      JCOL1=JCOL-1                                                      00004690
      DO 130 IROW=1,JCOL1                                               00004700
      AIJ=A(IROW,JCOL)                                                  00004710
      IF(DABS(AIJ)-THRSH)130,130,117                                    00004720
  117 AII=A(IROW,IROW)                                                  00004730
      AJJ=A(JCOL,JCOL)                                                  00004740
      S=AJJ-AII                                                         00004750
      IF(DABS(AIJ)-1.D-09*DABS(S))130,130,118                           00004760
  118 IFLAG=1                                                           00004770
      IF(1.D-10*DABS(AIJ)-DABS(S))116,119,119                           00004780
  119 S=.7071067811865475                                               00004790
      C=S                                                               00004800
      GO TO 120                                                         00004810
  116 T=AIJ/S                                                           00004820
      S=0.25/DSQRT(0.25+T*T)                                            00004830
      C=DSQRT(0.5+S)                                                    00004840
      S=2.*T*S/C                                                        00004850
  120 DO 121 I=1,IROW                                                   00004860
      T=A(I,IROW)                                                       00004870
      U=A(I,JCOL)                                                       00004880
      A(I,IROW)=C*T-S*U                                                 00004890
  121 A(I,JCOL)=S*T+C*U                                                 00004900
      I2=IROW+2                                                         00004910
      IF(I2-JCOL)127,127,123                                            00004920
  127 CONTINUE                                                          00004930
      DO 122 I=I2,JCOL                                                  00004940
      T=A(I-1,JCOL)                                                     00004950
      U=A(IROW,I-1)                                                     00004960
      A(I-1,JCOL)=S*U+C*T                                               00004970
  122 A(IROW,I-1)=C*U-S*T                                               00004980
  123 A(JCOL,JCOL)=S*AIJ+C*AJJ                                          00004990
      A(IROW,IROW)=C*A(IROW,IROW)-S*(C*AIJ-S*AJJ)                       00005000
      DO 124 J=JCOL,N                                                   00005010
      T=A(IROW,J)                                                       00005020
      U=A(JCOL,J)                                                       00005030
      A(IROW,J)=C*T-S*U                                                 00005040
  124 A(JCOL,J)=S*T+C*U                                                 00005050
```

```
      IF(NOYES)131,126,131                                                  00005060
  131 CONTINUE                                                              00005070
      DO 125 I=1,N                                                          00005080
      T=EIVR(I,IROW)                                                        00005090
      EIVR(I,IROW)=C*T-EIVR(I,JCOL)*S                                       00005100
  125 EIVR(I,JCOL)=S*T+EIVR(I,JCOL)*C                                       00005110
  126 CONTINUE                                                              00005120
      S=AIJ/ATOP                                                            00005130
      D=D-S*S                                                               00005140
      IF(D-DSTOP)1260,129,129                                               00005150
 1260 D=0.                                                                  00005160
      DO 128 JJ=2,N                                                         00005170
      DO 128 II=2,JJ                                                        00005180
      S=A(II-1,JJ)/ATOP                                                     00005190
  128 D=S*S+D                                                               00005200
      DSTOP=(1.D-06)*D                                                      00005210
  129 THRSH=DSQRT(D/AVGF)*ATOP                                              00005220
  130 CONTINUE                                                              00005230
      IF(IFLAG)115,134,115                                                  00005240
  134 T=A(1,1)                                                              00005250
      A(1,1)=EIVU(1)                                                        00005260
      EIVU(1)=T                                                             00005270
      DO 132 J=2,N                                                          00005280
      T=A(J,J)                                                              00005290
      A(J,J)=EIVU(J)                                                        00005300
      EIVU(J)=T                                                             00005310
      DO 132 I=2,J                                                          00005320
  132 A(I-1,J)=A(J,I-1)                                                     00005330
  133 RETURN                                                                00005340
      END                                                                   00005350
C                                                                           00005360
CC    ..................................................................... 00005370
CC                                                                          00005380
CC       SUBROUTINE INVRT                                                   00005390
CC                                                                          00005400
CC       PURPOSE                                                            00005410
CC          INVERT A MATRIX                                                 00005420
CC                                                                          00005430
CC       USAGE                                                              00005440
CC          CALL INVRT(A,N,D,L,M)                                           00005450
CC                                                                          00005460
CC       DESCRIPTION OF PARAMETERS                                          00005470
CC          A AND D MUST BE REAL*8                                          00005480
CC          A - INPJT MATRIX, DESTROYED IN COMPUTATION AND REPLACED BY      00005490
CC              RESULTANT INVERSE.                                          00005500
CC          N - ORDER OF MATRIX A                                           00005510
CC          D - RESULTANT DETERMINANT                                       00005520
C           L - WORK VECTOR OF LENGTH N                                     00005530
```

```
C           M - WORK VECTOR OF LENGTH N                                     00005540
CC                                                                          00005550
CC       REMARKS                                                            00005560
CC          MATRIX A MUST BE A GENERAL MATRIX                               00005570
CC                                                                          00005580
CC       SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                      00005590
CC          NONE                                                            00005600
CC                                                                          00005610
CC       METHOD                                                             00005620
CC          THE STANDARD GAUSS-JORDAN METHOD IS USED. THE DETERMINANT       00005630
CC          IS ALSO CALCULATED. A DETERMINANT OF ZERO INDICATES THAT        00005640
CC          THE MATRIX IS SINGULAR.                                         00005650
CC                                                                          00005660
CC    DOCUMENTATION:                                                        00005670
CC                  SUBROUTINE INVRT IS A COPY OF SUBROUTINE DMINV,         00005680
CC                  A CATALOGUED LIBRARY ROUTINE FROM THE W.R.CHURCH        00005690
CC                  COMPUTER CENTER, NAVPGSCOL, MONTEREY, CALIF. 93940      00005700
C     ..............................................................        00005710
C                                                                           00005720


      SUBROUTINE INVRT(A,N,D,L,M)                                           00005730
      DIMENSION A(1),L(1),M(1)                                              00005740
      DOUBLE PRECISION A,D,BIGA,HOLD                                        00005750
C                                                                           00005760
CC       ..........................................................         00005770
CCC                                                                         00005780
CCC      SEARCH FOR LARGEST ELEMENT                                         00005790
CC                                                                          00005800
      D=1.0D0                                                               00005810
      NK=-N                                                                 00005820
      DO 80 K=1,N                                                           00005830
      NK=NK+N                                                               00005840
      L(K)=K                                                                00005850
      M(K)=K                                                                00005860
      KK=NK+K                                                               00005870
      BIGA=A(KK)                                                            00005880
      DO 20 J=K,N                                                           00005890
      IZ=N*(J-1)                                                            00005900
      DO 20 I=K,N                                                           00005910
      IJ=IZ+I                                                               00005920
   10 IF(DABS(BIGA)-DABS(A(IJ))) 15,20,20                                   00005930
   15 BIGA=A(IJ)                                                            00005940
      L(K)=I                                                                00005950
      M(K)=J                                                                00005960
   20 CONTINUE                                                              00005970
C                                                                           00005980
C        INTERCHANGE ROWS                                                   00005990
```

```
C                                                                       00006000
      J=L(K)                                                            00006010
      IF(J-K) 35,35,25                                                  00006020
   25 KI=K-N                                                            00006030
      DO 30 I=1,N                                                       00006040
      KI=KI+N                                                           00006050
      HOLD=-A(KI)                                                       00006060
      JI=KI-K+J                                                         00006070
      A(KI)=A(JI)                                                       00006080
   30 A(JI) =HOLD                                                       00006090
C                                                                       00006100
CC         INTERCHANGE COLUMNS                                          00006110
C                                                                       00006120
   35 I=M(K)                                                            00006130
      IF(I-K) 45,45,38                                                  00006140
   38 JP=N*(I-1)                                                        00006150
      DO 40 J=1,N                                                       00006160
      JK=NK+J                                                           00006170
      JI=JP+J                                                           00006180
      HOLD=-A(JK)                                                       00006190
      A(JK)=A(JI)                                                       00006200
   40 A(JI) =HOLD                                                       00006210
C                                                                       00006220
CC         DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT ELEMENT IS      00006230
CC         CONTAINED IN BIGA)                                           00006240
C                                                                       00006250
   45 IF(BIGA) 48,46,48                                                 00006260
   46 D=0.0D0                                                           00006270
      RETURN                                                            00006280
   48 DO 55 I=1,N                                                       00006290
      IF(I-K) 50,55,50                                                  00006300
   50 IK=NK+I                                                           00006310
      A(IK)=A(IK)/(-BIGA)                                               00006320
   55 CONTINUE                                                          00006330
C                                                                       00006340
CC         REDUCE MATRIX                                                00006350
C                                                                       00006360
      DO 65 I=1,N                                                       00006370
      IK=NK+I                                                           00006380
      HOLD=A(IK)                                                        00006390
      IJ=I-N                                                            00006400
      DO 65 J=1,N                                                       00006410
      IJ=IJ+N                                                           00006420
      IF(I-K) 60,65,60                                                  00006430
   60 IF(J-K) 62,65,62                                                  00006440
   62 KJ=IJ-I+K                                                         00006450
      A(IJ)=HOLD*A(KJ)+A(IJ)                                            00006460
   65 CONTINUE                                                          00006470
```

```
C                                                                       00006480
CC         DIVIDE ROW BY PIVOT                                          00006490
C                                                                       00006500
      KJ=K-N                                                            00006510
      DO 75 J=1,N                                                       00006520
      KJ=KJ+N                                                           00006530
      IF(J-K) 70,75,70                                                  00006540
   70 A(KJ)=A(KJ)/BIGA                                                  00006550
   75 CONTINUE                                                          00006560
C                                                                       00006570
CC         PRODUCT OF PIVOTS                                            00006580
C                                                                       00006590
      D=D*BIGA                                                          00006600
C                                                                       00006610
CC         REPLACE PIVOT BY RECIPROCAL                                  00006620
C                                                                       00006630
      A(KK)=1.0/BIGA                                                    00006640
   80 CONTINUE                                                          00006650
C                                                                       00006660
CC         FINAL ROW AND COLUMN INTERCHANGE                             00006670
C                                                                       00006680
      K=N                                                               00006690
  100 K=(K-1)                                                           00006700
      IF(K) 150,150,105                                                 00006710
  105 I=L(K)                                                            00006720
      IF(I-K) 120,120,108                                               00006730
  108 JQ=N*(K-1)                                                        00006740
      JR=N*(I-1)                                                        00006750
      DO 110 J=1,N                                                      00006760
      JK=JQ+J                                                           00006770
      HOLD=A(JK)                                                        00006780
      JI=JR+J                                                           00006790
      A(JK)=-A(JI)                                                      00006800
  110 A(JI) =HOLD                                                       00006810
  120 J=M(K)                                                            00006820
      IF(J-K) 100,100,125                                               00006830
  125 KI=K-N                                                            00006840
      DO 130 I=1,N                                                      00006850
      KI=KI+N                                                           00006860
      HOLD=A(KI)                                                        00006870
      JI=KI-K+J                                                         00006880
      A(KI)=-A(JI)                                                      00006890
  130 A(JI) =HOLD                                                       00006900
      GO TO 100                                                         00006910
  150 RETURN                                                            00006920
      END                                                               00006930
```

```
      SUBROUTINE ANSER                                                      00006940
      IMPLICIT REAL*8(A-H,O-Z)                                              00006950
      COMMON/INT/NCON(50,4),L1(6),L2(6),NEL,NJT,NEQ,NEID(50),JID(100),IO    00006960
      COMMON/FLP/BK(200,6),BKT(6,200),R(200),COORD(100,2),SS(100,6),SN(     00006970
     1100,6),SK(8,6),BSYM(6,6),ASYM(6,6),U(100),V(100),BR(6),C(6),A,B,      00006980
     1TITLE(20),T                                                           00006990
C     ***************************************************************** 00007000
C                                                                           00007010
C     SUBROUTINE ANSER OUTPUTS THE SIX MATERIAL CONSTANTS.  THE VECTOR      00007020
C     OF MATERIAL CONSTANTS IS GENERATED BY MULTIPLICATION OF THE VECTOR    00007030
C     BR BY THE INVERSE OF BSYM.                                            00007040
C                                                                           00007050
C     ***************************************************************** 00007060
      DO 10 J=1,6                                                           00007070
   10 C(J)=0.000                                                            00007080
      DO 20 K=1,6                                                           00007090
      DO 20 L=1,6                                                           00007100
   20 C(K)=C(K)+BSYM(K,L)*BR(L)                                             00007110
      WRITE(6,50)                                                           00007120
   50 FORMAT('1','THE VECTOR OF MATERIAL CONSTANTS',                        00007130
     1//,' TRANSPOSE<C11,C12,C13,C22,C23,C33>=')                            00007140
      DO 60 I=1,6                                                           00007150
   60 WRITE(6,70) C(I)                                                      00007160
   70 FORMAT('0',15X,1G24.16)                                               00007170
      RETURN                                                                00007180
      END                                                                   00007190
```

REPRESENTATIVE OUTPUT: 15 KIP LOAD, 4-NODES. (ELASTIC)

```
NUMBER OF ELEMENTS IN STRUCTURE          4
NUMBER OF JOINTS IN STRUCTURE            9
THICKNESS IS(UNIFORM)=  0.10000000000000000D 00

CONNECTIVITY MATRIX

     EL.ID     I-J-K-L CONNECTIVITY
         1     2     3     6     5
         2     5     6     9     8
         3     1     2     5     4
         4     4     5     8     7

  JOINT ID              X-COORDINATE              Y-COORDINATE
         1   10.000000000000000        0.0
         2   5.0000000000000000        0.0
         3   0.0                       0.0
         4   10.000000000000000        -5.000000000000000
         5   5.0000000000000000        -5.000000000000000
         6   0.0                       -5.000000000000000
         7   10.000000000000000        -10.00000000000000
         8   5.0000000000000000        -10.00000000000000
         9   0.0                       -10.00000000000000

                     HORIZ. FORCE              VERT. FORCE
          0.9103955176022619         3.924156487558248
          0.0                        7.151687024883492
         -0.9103955176022620         3.924156487558250
          0.0                       0.0
          0.0                       0.0
          0.0                       0.0
          0.0                       -5.000000000000000
          0.0                       -5.000000000000000
          0.0                       -5.000000000000000

  JOINT ID            X-DISPLACEMENT            Y-DISPLACEMENT
         1  0.0                       0.0
         2  0.0                       0.0
         3  0.0                       0.0
         4 -0.4929293242488149D-03 -0.2617793655895807D-02
         5  0.2014108968973132D-18 -0.2189597399146653D-02
         6  0.4929293242488153D-03 -0.2617793655895808D-02
         7 -0.1315568119229649D-02 -0.5164481934212524D-02
         8  0.4872528994099446D-18 -0.3827868226091310D-02
         9  0.1315568119229649D-02 -0.6164481934212524D-02

EIGEN VALUES OF BSYM=BKT*BK
            0.2031967468551705D-08
            0.2443927086398269D-06
            0.3938446247547611D-07
            0.1542630641666066D-06
            0.3839002888303476D-06
            0.3825140881455582D-07

THE VECTOR OF MATERIAL CONSTANTS

TRANSPOSE<C11,C12,C13,C22,C23,C33>=

                  32147.99848740955

                  9226.475565886535

                -0.5826520139618010D-11

                  32147.99848740942

                -0.3403485991231466D-11

                  11460.76146076146
```

BIBLIOGRAPHY

1. Abramowitz, M. and I. A. Segun (eds.), "2500 Five Digit Random Numbers", in Handbook of Mathematical Functions, Dover, New York, 1968.

2. Bychawski, Z. and A. Fox, "The Constitutive Equation of a Complete Nonlinear Viscoelastic Material", Acta Mechanica, 6, 2-3, 1968.

3. Cantin, G., Naval Postgraduate School, Monterey, California, private communication (1972).

4. Clough, R. W., "The Finite Element Method in Structural Mechanics", in Stress Analysis, O. C. Zienkiewicz and G. S. Holister (eds), Wiley, London, 1965.

5. Desai, C. S. and J. F. Abel, Introduction to the Finite Element Method, Van Nostrand-Reinhold, New York, 1972.

6. Dong, R., K. Pister, and R. Dunham, "Mechanical Characterization of Non-linear Viscoelastic Solids for Iterative Solutions of Boundary Value Problems", Acta Mechanica, 9, 1-2, 1970.

7. Evans, R. J. and K. Pister, "Constitutive Equations for a Class of Non-linear Elastic Solids", International Journal of Solids and Structures, 2, 4, 1966.

8. Filonenko-Borodich, M. M., Theory of Elasticity, (M. Konayeva, translator), Noordhoff, Groningen (The Netherlands), undated.

9. Gleason, R. F., "On the Determination of Material Characteristics in Nonlinear Viscoelasticity", (U), U. S. Air Force Rocket Propulsion Laboratory, Stanford University, AFRPL-TR-67-192(1967).

10. Hearmon, R. F. S., An Introduction to Applied Anisotropic Elasticity, Oxford University, Oxford, 1961.

11. Jaensson, B. O. and B. O. Sundström, "Determination of Young's Modulus and Poisson's Ratio for WC-Co Alloys by the Finite Element Method", Materials Science and Engineering, 9, 4, 1972.

12. Kavanagh, K. T., The Finite Element Analysis of Physically and Kinematically Nonlinear Elastic Solids, Ph.D. dissertation, University of California (Berkeley), Berkeley, California (1969).

13. Kavanagh, K. T., and R. W. Clough, "Finite Element Applications in the Characterization of Elastic Solids", International Journal of Solids and Structures, 7, 1971.

14. Muskhelishvili, N. I., Some Basic Problems of the Mathematical Theory of Elasticity, (J. R. M. Radok, translator), Noordhoff, Groningen (The Netherlands), 1953.

15. Nachlinger, R. R. and H. H. Calvit, "On Approximate Constitutive Equations in Nonlinear Viscoelasticity", Acta Mechanica, 7, 4, 1969.

16. Oden, J. T., "Numerical Formulations of Non-linear Elasticity Problems", Journal of the American Society of Civil Engineers (Structural), 93, 3, 1967.

17. Oden, J. T., "A General Theory of Finite Elements", International Journal of Numerical Methods in Engineering, 1, 1969.

18. Sandhu, R. S. and K. S. Pister, "Variational Principles for Boundary Value and Initial-boundary Value Problems in Continum Mechanics", International Journal of Solids and Structures, 7, 1971.

19. Timoshenko, S. and J. N. Goodier, Theory of Elasticity, McGraw-Hill, New York, 1951.

20. Washizu, K., Variational Methods in Elasticity and Plasticity, Pergamon, Oxford, 1968.

21. Zienkiewicz, O. C., The Finite Element Method in Engineering Science, McGraw-Hill, London, 1971.

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13. ABSTRACT

Finite element methods are applied to the problem of characterizing linear, anisotropic elastic solids. The conventional finite element displacement formulation is used to simulate an elastic material in plane stress. An inverted finite element formulation is then applied, and the characterizing six material constants are calculated as numerical results.

A possible test device for the experimental characterization of anisotropic solids is postulated, the precision of displacement measurements to be required for such a device being determined by random perturbation analysis. Numerical constants accurate to within three percent are predicted if a precision of one part in eight hundred (1/800) can be measured. Numerical constants accurate to within one percent are predicted if a precision of one part in eight thousand (1/8000) can be measured in the test device.

KEY WORDS	LINK A		LINK B		LINK C	
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